STRONG q-CONVEXITY IN UNIFORM NEIGHBORHOODS OF SUBVARIETIES IN COVERINGS OF COMPLEX SPACES

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Abstract. The main result is that, for any projective compact analytic subset Y of dimension $q > 0$ in a reduced complex space X, there is a neighborhood Ω of Y such that, for any covering space $\Upsilon\colon \widehat{X} \to X$ in which $\widehat{Y} \equiv \Upsilon^{-1}(Y)$ has no noncompact connected analytic subsets of pure dimension q with only compact irreducible components, there exists a C^∞ exhaustion function φ on $\widehat{X}$ which is strongly q-convex on $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$ outside a uniform neighborhood of the q-dimensional compact irreducible components of $\widehat{Y}$.

Introduction

According to the main result of [Fra], for any projective compact analytic subset Y of dimension $q > 0$ in a complex manifold X, there exists a neighborhood Ω of Y in X such that, for any covering space $\Upsilon\colon \widehat{X} \to X$ in which $\widehat{Y} \equiv \Upsilon^{-1}(Y)$ has no compact irreducible components, there exists a C^∞ exhaustion function φ on $\widehat{X}$ which is strongly q-convex on $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$. The case $n = 1$ was obtained in [N] and [Co2], and a similar result was first obtained in [Co-V] for Y a fiber of a suitable proper holomorphic mapping (see also [Mi]). The main goal of this paper is a more general version in which X is only a reduced complex space and $\widehat{Y}$ may have some compact irreducible components (but no infinite chains of q-dimensional compact irreducible components).

Let Ω be an open subset of a reduced complex space X and let q be a positive integer. A function φ on Ω is C^∞ *strongly q-convex* if, for each point $p \in \Omega$, there is a proper holomorphic embedding Φ of a neighborhood U of p in Ω into an open set $U' \subset \mathbb{C}^N$ and a function $\varphi' \in C^\infty(U')$ such that $\varphi' \circ \Phi = \varphi$ on U and such that the Levi-form $\mathcal{L}(\varphi')$ has at most $q - 1$ nonpositive eigenvalues at each point. We also say that φ is of class $\mathcal{SP}^\infty(q)$ and we write $\varphi \in \mathcal{SP}^\infty(q)(\Omega)$. For $q = 1$, we also say that the function is C^∞ *strictly plurisubharmonic.*

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There does not seem to be a single best analogous notion of weak q-convexity. One natural analogue is the following. We will say that a function φ on Ω is of class $\mathcal{W}^\infty(q)$ if, for each point $p \in \Omega$, there is a proper holomorphic embedding Φ of a neighborhood U of p in Ω into an open set $U' \subset \mathbb{C}^N$ and a function $\varphi' \in C^\infty(U')$ such that $\varphi' \circ \Phi = \varphi$ on U and such that the Levi-form $\mathcal{L}(\varphi')$ has at most $q-1$ negative eigenvalues at each point. For $q = 1$, we also say that the function is C^∞ *plurisubharmonic.*

Since every local embedding factors through the Zariski tangent space, it follows that, if $\varphi \in \mathcal{W}^\infty(q)(\Omega)$ ($\mathcal{SP}^\infty(q)(\Omega)$), then, for every point $p \in \Omega$ and every local holomorphic model (U, Φ, U') with $p \in U$, there is a neighborhood V' of $\Phi(p)$ in U' and a function $\varphi' \in \mathcal{W}^\infty(q)(V')$ (respectively, $\mathcal{SP}^\infty(q)(V')$) with $\varphi = \varphi' \circ \Phi$ on $\Phi^{-1}(V')$.

The main result of this paper is the following (see also the stronger version Theorem 7.1).

Theorem 0.1. *Let X be a connected reduced complex space and let Y be a compact analytic subset of dimension $q > 0$ in X. Assume that Y admits a projective embedding. Then there exists a neighborhood Ω of Y in X and a discrete subset F of $\mathbb{R}$ such that, for any connected covering space $\Upsilon\colon \widehat{X} \to X$ in which $\widehat{Y} \equiv \Upsilon^{-1}(Y)$ has no noncompact connected analytic subsets of pure dimension q with only compact irreducible components, there exists a C^∞ exhaustion function φ on $\widehat{X}$ which is of class $\mathcal{W}^\infty(q)$ on $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$ and of class $\mathcal{SP}^\infty(q)$ on $\widehat{\Omega} \setminus \varphi^{-1}(F)$.*

Remarks. 1. In place of Y being projective, we need only assume that Y admits a nowhere dense analytic subset S containing Y_{sing} such that $Y \setminus S$ is Stein.

2. In the proof for X a 2-dimensional complex manifold (and $\dim Y = 1$) in [N], the second author mistakenly only wrote the statement and proof for Galois coverings. In the 2-dimensional case, the statement and proof are easily modified to give the version for general coverings. In higher dimensions, more care must be taken in dealing with the singular set of Y (which need not be discrete).

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1. The Zariski tangent space and Hermitian metrics

In this section we recall notions of tangent vectors and Hermitian metrics on complex spaces. Throughout this section X (or $(X, \mathcal{O}_X)$) will denote a reduced complex space.

Analytic subsets. Unless otherwise indicated, by an *analytic subset* of X we will mean a properly embedded reduced analytic subspace A; that is, a closed subset which is locally

the zero set of a collection of holomorphic functions together with structure sheaf given by the restrictions of local holomorphic functions in X. We will denote the structure sheaf of A by $\mathcal{O}_A$ and the ideal sheaf by $\mathcal{I}_A$.

By a *local holomorphic model* (or simply a *local model*) in X we will mean a triple (U, Φ, U'), where U is an open subset of X, U' is an open subset of $\mathbb{C}^N$ for some N, and $\Phi\colon U \to U'$ is a holomorphic map which maps U isomorphically onto an analytic subset $\Phi(U)$ of some open subset of U'. Note that we do *not* require Φ to be proper, but it will be convenient for our purposes to also have an open set U' containing $\Phi(U)$ to which to refer. If Φ is a proper embedding, then we will call (U, Φ, U') a *proper local holomorphic model* (or simply a *proper local model*). Observe that, for (U, Φ, U') to be a local model, we do require in general that Φ properly embed U into *some* open subset of U' (i.e. the topology on U induced by U' agrees with the original topology on U).

The Zariski tangent linear space. For each point $p \in X$, the vector space

$$\mathcal{T}^{(1)}_p X \equiv (\mathfrak{m}_p/\mathfrak{m}_p^2)^*,$$

where $\mathfrak{m}_p$ is the maximal ideal in $\mathcal{O}_X$ at p, is called the *Zariski tangent space at p*. The *Zariski tangent linear space*

$$\Pi_{\mathcal{T}^{(1)}X}\colon \mathcal{T}^{(1)}X \equiv \bigcup_{p\in X} \mathcal{T}^{(1)}_p X \to X$$

has a natural reduced complex analytic linear space structure with the following local models. Given a proper local model (U, Φ, U') in X with $U' \subset \mathbb{C}^N$, setting $A = \Phi(U)$ we get the analytic subset

$$B \equiv \left\{ w \in \mathcal{T}^{(1)}U' \mid z = \Pi_{\mathcal{T}^{(1)}U'}(w) \in A \text{ and } df(w) = 0 \text{ for every } f \in (\mathcal{I}_A)_z \right\}$$

of $\mathcal{T}^{(1)}U' = T^{1,0}U' = U' \times \mathbb{C}^N$ and a bijection $\Phi_*\colon \mathcal{T}^{(1)}U = \mathcal{T}^{(1)}X \restriction_U = {\Pi_{\mathcal{T}^{(1)}X}}^{-1}(U) \to B$ determined by

$$dh(\Phi_* v) = v([h\circ\Phi - h(\Phi(p))]_p) \qquad \forall\, h \in (\mathcal{O}_{\mathbb{C}^N})_{\Phi(p)}, \quad p \in U, \quad v \in \mathcal{T}^{(1)}_p U.$$

For each point $p \in U$, the map $(\Phi_*)_p = \Phi_* \restriction_{\mathcal{T}^{(1)}_p X}\colon \mathcal{T}^{(1)}_p X \to \{\Phi(p)\} \times \mathbb{C}^N$ is an injective complex linear map. The triple $([\Pi_{\mathcal{T}^{(1)}X}]^{-1}(U), \Phi_*, U' \times \mathbb{C}^N)$ is the proper local holomorphic model in $\mathcal{T}^{(1)}X$ corresponding to the proper local holomorphic model (U, Φ, U') in X. Observe that we may identify B with $\mathcal{T}^{(1)}A$ and we get a commutative diagram of holomorphic maps

$$\begin{array}{ccc} \mathcal{T}^{(1)}U & \xrightarrow{\Phi_*} & \mathcal{T}^{(1)}A \\ \downarrow & & \downarrow \\ U & \xrightarrow{\Phi} & A \end{array}$$

in which Φ and Φ_* are isomorphisms. Note that $\Pi_{\mathcal{T}^{(1)}X} : \mathcal{T}^{(1)}X \to X$ need *not* be an open mapping if X is singular.

We will denote by $TX = \bigcup_{p\in X} T_pX$ the real analytic linear space associated to the complex analytic linear space $\mathcal{T}^{(1)}X$ and by $J\colon TX \to TX$ the corresponding complex structure. We may identify $\mathcal{T}^{(1)}X = (TX, J)$ with the $(1,0)$ part $T^{1,0}X$ of the complexification of TX under the isomorphism $v \mapsto \frac{1}{2}(v - \sqrt{-1}Jv)$.

Tangent mappings. A holomorphic mapping $\Psi\colon X \to Y$ of reduced complex spaces X and Y induces a holomorphic mapping $\Psi_*\colon \mathcal{T}^{(1)}X \to \mathcal{T}^{(1)}Y$ such that, for each $p \in X$, the restriction $(\Psi_*)_p = \Psi_* \restriction_{\mathcal{T}^{(1)}_p X}\colon \mathcal{T}^{(1)}_p X \to \mathcal{T}^{(1)}_{\Psi(p)} Y$ is the linear map given by

$$(\Psi_*)_p(v)([f]_{\Psi(p)}) = v([f\circ\Psi]_p) \qquad \forall\, v \in \mathcal{T}^{(1)}_p X,\ f \in \mathfrak{m}_{Y,\Psi(p)}.$$

If R is a function on $\mathcal{T}^{(1)}Y$, then we denote the function $R\circ\Psi_*$ on $\mathcal{T}^{(1)}X$ by Ψ^*R (instead of $(\Psi_*)^*R$). If $(\Psi_*)_p$ is injective for some point $p \in X$, then the restriction of Ψ to a small neighborhood U of p in X embeds U properly into some neighborhood of $\Psi(p)$ in Y. Moreover, Ψ_* is a (proper) embedding if and only if Ψ is a (proper) embedding. Finally, given another holomorphic mapping $\Phi\colon Y \to Z$ to a reduced complex space Z, we have $(\Phi\circ\Psi)_* = \Phi_*\circ\Psi_*$.

Similarly, a C^∞ mapping $\Psi\colon M \to X$ of a real C^∞ manifold M into X induces a C^∞ tangent mapping $\Psi_*\colon TM \to TX$. For Ψ a holomorphic mapping, the above mappings correspond under the isomorphism $(TX, J) \cong \mathcal{T}^{(1)}X$.

Embedding dimension. For each point $p \in X$, there is a proper local holomorphic model $\Phi\colon U \to U' \subset \mathcal{T}^{(1)}_p X \cong \mathbb{C}^d$ on a neighborhood U of p with $\Phi(p) = 0$ and $\Phi_*\left(\mathcal{T}^{(1)}_p X\right) = \mathcal{T}^{(1)}_0 \Phi(U) = \{0\}\times\mathbb{C}^d$. Furthermore, every holomorphic embedding of a neighborhood of p into a complex manifold factors into the composition with Φ of an embedding of a neighborhood of 0. In other words, if $\Psi\colon V \to Y$ is a holomorphic embedding of a neighborhood $V \subset U$ of p into a complex manifold Y and Ψ' is any holomorphic lifting of Ψ to a neighborhood V' of 0 in $\mathbb{C}^d$ (i.e. $\Psi'\colon V' \to Y$ is a holomorphic map with $\Psi'\circ\Phi = \Psi$ on a neighborhood of p), then Ψ' embeds a neighborhood of $0 \in \mathbb{C}^d$ into a neighborhood of $\Psi(p)$ in Y. In particular, $d = \dim \mathcal{T}^{(1)}_p X = \operatorname{embdim}_p X$ is the (minimal) embedding dimension of the complex analytic space germ at p determined by X.

The associated direct sum spaces. For a positive integer q, we will denote by $\mathcal{T}^{(q)}X$ the q-fold reduced fiber product space

$$\Pi_{\mathcal{T}^{(q)}X}\colon \mathcal{T}^{(q)}X = \mathcal{T}^{(1)}X \times_{\Pi_{\mathcal{T}^{(1)}X}} \cdots \times_{\Pi_{\mathcal{T}^{(1)}X}} \mathcal{T}^{(1)}X \to X.$$

Thus, for each point $p \in X$,

$$(\Pi_{\mathcal{T}^{(q)}X})^{-1}(p) = \mathcal{T}^{(q)}_pX = \mathcal{T}^{(1)}_pX \times \cdots \times \mathcal{T}^{(1)}_pX = \mathcal{T}^{(1)}_pX \oplus \cdots \oplus \mathcal{T}^{(1)}_pX \quad (q \text{ summands}).$$

If $\Psi\colon X \to Y$ is a holomorphic mapping of reduced complex spaces, then, for every positive integer q, we define the holomorphic mapping (linear on fibers) $\Psi^{(q)}_*\colon \mathcal{T}^{(q)}X \to \mathcal{T}^{(q)}Y$ by $(v_1, \dots, v_q) \mapsto (\Psi_*v_1, \dots, \Psi_*v_q)$. If R is a function on $\mathcal{T}^{(q)}Y$, then we define the function Ψ^*R on $\mathcal{T}^{(q)}X$ by $\Psi^*R = R \circ \Psi^{(q)}_*$.

Hermitian metrics. A *Hermitian metric* g in X is a C^∞ function $g\colon \mathcal{T}^{(2)}X \to \mathbb{C}$ such that $g \restriction_{\mathcal{T}^{(2)}_pX}$ is a Hermitian inner product for each point $p \in X$. Equivalently (see, for example, [JNR]), for each point $p \in X$, there is a local model (U, Φ, U') in X with $p \in U$ and a (C^∞) Hermitian metric g' in U' such that $g \restriction_{\mathcal{T}^{(2)}U} = \Phi^*g'$. We will call g' a *local representation for* g. If $\Psi\colon Y \to X$ is a local holomorphic embedding, then we get a pullback Hermitian metric Ψ^*g in Y. We will also write $g \restriction_Y = \iota^*g = g \restriction_{\mathcal{T}^{(2)}Y}$ for an inclusion $\iota\colon Y \subset X$.

Remarks. 1. Since every local model in a neighborhood of a point $p \in X$ factors through $\mathcal{T}^{(1)}_pX$ in a neighborhood of p, with both factors embeddings, it follows that a local representation g' as above exists in a neighborhood of p in *any* local model.
2. The Hermitian metric g also determines a Hermitian metric and an associated Riemannian metric in the isomorphic complex linear space (TX, J).

Distance in a complex space. Assume that X is connected and let g be a Hermitian metric in X. Given a piecewise C^∞ curve $\gamma\colon [a, b] \to X$, we define the *length of* γ *with respect to* g by

$$\ell_g(\gamma) = \int_a^b |\dot\gamma(t)|_g \, dt.$$

Given two points, $p, q \in X$, we define the *distance between* p *and* q *with respect to* g by

$$\mathrm{dist}_g(p, q) = \inf \{ \ell_g(\gamma) \mid \gamma \text{ is a piecewise } C^\infty \text{ curve from } p \text{ to } q \text{ in } X \}.$$

This distance is finite since such a piecewise C^∞ path connecting two given points always exists (for example, by the existence of a resolution of singularities). We have the following standard fact:

Proposition 1.1. *Let X be connected and let g be a Hermitian metric in X. Then $\mathrm{dist}_g(\cdot,\cdot)$ is a metric (in the sense of a distance function) in X which induces the given complex space topology in X. Furthermore, there exists a positive continuous function α on X such that, if β is any positive C^∞ function on X with $\beta \geq \alpha$ on the complement of some compact subset of X and $h = \beta \cdot g$, then, for each point $p \in X$, the function $x \mapsto \mathrm{dist}_h(x,p)$ is an exhaustion function. In particular, dist_h is a complete metric.*

Proof. It is easy to see that dist_g is symmetric and nonnegative on $X \times X$ and zero on the diagonal. The triangle inequality is also easily verified. Given a point $a \in X$, we may choose a proper local holomorphic model (U, Φ, U') with $a \in U$ and a Hermitian metric g' on U' with $\Phi^* g' = g$ on U. Given a connected relatively compact neighborhood V of a in U, we may choose a connected relatively compact neighborhood V' of $\Phi(a)$ in U' with $\Phi^{-1}(V') = V$. Let $r = \mathrm{dist}_{g'}(\Phi(a), U' \setminus V') > 0$. Given a point $x \in X \setminus V$ and a piecewise C^∞ path γ in X from a to x, there is some $t \in (0,1]$ with $\gamma([0,t)) \subset V$ and $\gamma(t) \in \partial V$. Thus $\ell_g(\gamma) \geq \ell_{g'}\left(\Phi\left(\gamma\restriction_{[0,t]}\right)\right) \geq r$. It follows that $\mathrm{dist}_g(a,\cdot) > 0$ on $X \setminus \{a\}$ (since, for any given $x \in X \setminus \{a\}$, we may choose such a neighborhood V not containing x) and $B_g(a;r) \subset V$. Thus dist_g is a metric inducing a topology which is finer than the given topology.

For the reverse containment, let $\Psi\colon \check{X} \to X$ be a resolution of singularities. Thus $\check{X}$ is a smooth complex space with connected components $\{\check{X}_i\}_{i\in I}$, Ψ is a surjective proper holomorphic map, the analytic set $E = \Psi^{-1}(X_{\mathrm{sing}})$ is nowhere dense in $\check{X}$, Ψ maps $\check{X} \setminus E$ isomorphically onto X_{reg}, and the distinct irreducible components of X are given by $X_i = \Psi(\check{X}_i)$ for $i \in I$. We may also choose a Hermitian metric h on $\check{X}$, and we may let $\check{g}$ be the Hermitian metric given by $\check{g} = \Psi^* g + h$. Given a point $a \in X$ and a constant $r > 0$, we may choose a neighborhood U of the compact analytic set $A = \Psi^{-1}(a)$ in $\check{X}$ such that, for each $i \in I$, $U \cap \check{X}_i$ is contained in the r-neighborhood of $A \cap \check{X}_i$ with respect to the metric $\mathrm{dist}_{\check{g}\restriction_{\check{X}_i}}$ (in particular, $U \cap \check{X}_i = \emptyset$ if $A \cap \check{X}_i = \emptyset$). The set $\Psi(U)$ contains a neighborhood V of a in X and, given a point $\check{x} \in U$, we may choose a C^∞ path $\check{\gamma}$ in $\check{X}_i$ for some $i \in I$ with $\check{\gamma}(0) \in A$, $\check{\gamma}(1) = \check{x}$, and $\ell_{\check{g}}(\check{\gamma}) < r$. The path $\gamma = \Psi(\check{\gamma})$ from a to $x = \Psi(\check{x})$ then satisfies

$$\ell_g(\gamma) = \int_0^1 |\dot{\gamma}(t)|_g \, dt \leq \int_0^1 |\dot{\check{\gamma}}(t)|_{\check{g}} \, dt < r.$$

Thus $a \in V \subset \Psi(U) \subset B_g(a;r)$ and hence the metric topology and the given topology are equal.

Finally, for the construction of the function α, we may assume without loss of generality that X is noncompact. We may choose a sequence of domains $\{\Omega_\nu\}_{\nu=0}^\infty$ in X such that $X = \bigcup_{\nu=0}^\infty \Omega_\nu$ and $\Omega_{\nu-1} \Subset \Omega_\nu$ for each $\nu = 1, 2, 3, \dots$. There then exists a continuous function $\alpha\colon X \to (1,\infty)$ such that $\alpha > [\mathrm{dist}_g(\Omega_{\nu-1}, X \setminus \Omega_\nu)]^{-2}$ on $X \setminus \Omega_{\nu-1}$ for each $\nu = 1, 2, 3, \dots$. Suppose $p \in X$, β is a positive C^∞ function on X with $\beta \geq \alpha$ on the complement of some compact set $K \subset X$, and $h = \beta \cdot g$. We may fix $\mu \in \mathbb{N}$ with $\{p\} \cup K \subset \Omega_{\mu-1}$. Suppose $R > 0$ and $\nu \geq R + \mu$. If γ is a piecewise C^∞ path from p to a point $x \in X \setminus \Omega_\nu$, then we have numbers

$$0 < s_\mu < t_\mu \leq s_{\mu+1} < t_{\mu+1} \leq s_{\mu+2} < t_{\mu+2} \leq \cdots \leq s_\nu < t_\nu \leq 1$$

such that, for $j = \mu, \dots, \nu$, we have $\gamma((s_j, t_j)) \subset \Omega_j \setminus \Omega_{j-1}$, $\gamma(s_j) \in \partial\Omega_{j-1}$, and $\gamma(t_j) \in \partial\Omega_j$. Hence

$$\ell_h(\gamma) \geq \sum_{j=\mu}^{\nu} \int_{s_j}^{t_j} |\dot\gamma(t)|_h \, dt \geq \sum_{j=\mu}^{\nu} [\mathrm{dist}_g(\Omega_{j-1}, X \setminus \Omega_j)]^{-1} \int_{s_j}^{t_j} |\dot\gamma(t)|_g \, dt \geq \nu - \mu + 1 > R.$$

Thus $B_h(p; R) \subset \Omega_\nu \Subset X$ and hence the function $x \mapsto \mathrm{dist}_h(x, p)$ exhausts X. $\square$

To close this section, we record two facts for later use.

Lemma 1.2. *Suppose X is connected, g is a Hermitian metric on X, and $x \mapsto \mathrm{dist}_g(x,p)$ is an exhaustion function on X for some (hence, for each) point $p \in X$. Then we have the following:*

(a) *For each point $p \in X$ and each constant $R > 0$, the set*

$$K(p,R) \equiv \{\, [\alpha] \in \pi_1(X,p) \mid \alpha \text{ is a piecewise } C^\infty \text{ loop in } X \text{ of length } < R \,\}$$

is finite.

(b) *For every connected covering space $\Upsilon\colon \widehat{X} \to X$, $x \mapsto \mathrm{dist}_{\hat g}(x,p)$ is an exhaustion function for each point $p \in \widehat{X}$, where $\hat g = \Upsilon^* g$. In particular, $\mathrm{dist}_{\hat g}(\cdot,\cdot)$ is a complete metric on $\widehat{X}$.*

Proof. For the proof of (a), we fix a point $p \in X$ and number $r > 0$ so small that, for each point a in the compact set $D \equiv \overline{B_g(p;R)}$, the ball $B_g(a; 3r)$ is contained in some contractible open set. We may also choose points $p = p_1, p_2, \dots, p_k \in D$ such that the balls $B_1 = B_g(p_1; r), \dots, B_k = B_g(p_k; r)$ form a covering for D, a Lebesgue number $\delta > 0$ for this covering with $\delta < r$, and a positive integer m such that $R/m < \delta$. For each pair of indices i, j, we may choose a piecewise C^∞ path $\lambda_{ij} = \lambda_{ji}^{-1}$ from p_i to p_j such that $\ell_g(\lambda_{ij}) < \mathrm{dist}_g(p_i, p_j) + r$. Now any piecewise C^∞ loop α of length $< R$ based at p is

homotopic to a loop $\alpha_1 * \alpha_2 * \cdots * \alpha_m$ in D; where, for each $\nu = 1, \dots, m$, α_ν is a piecewise C^∞ path of length $< \delta$ and is, therefore, contained in B_{i_ν} for some index i_ν. We may choose $i_1 = i_m = 1$. For each $\nu = 2, \dots, m$, we may choose a piecewise C^∞ path ρ_ν of length $< r$ from $\alpha_{\nu-1}(1) = \alpha_\nu(0)$ to $p_{i_{\nu-1}}$. Thus $\alpha \sim \gamma_1 * \cdots * \gamma_m$, where $\gamma_1 = \alpha_1 * \rho_2$, $\gamma_m = \rho_m^{-1} * \alpha_m$, and, for $\nu = 2, \dots, m-1$, $\gamma_\nu = \rho_\nu^{-1} * \alpha_\nu * \rho_{\nu+1}$. For each $\nu = 2, \dots, m$, we have $\ell_g(\gamma_\nu) < 3r$. We also have $\alpha_\nu(0) \in B_{i_{\nu-1}} \cap B_{i_\nu}$, so $\ell_g\left(\lambda_{i_{\nu-1}i_\nu}\right) < \operatorname{dist}_g(p_{i_{\nu-1}}, p_{i_\nu}) + r < 3r$. Therefore, γ_ν is homotopic to $\lambda_{i_{\nu-1}i_\nu}$. Moreover, γ_1 is homotopic to the trivial loop at $p_1 = p$. Thus α is homotopic to the loop $\lambda_{i_1i_2} * \lambda_{i_2i_3} * \cdots * \lambda_{i_{m-1}i_m}$ and the claim follows.

For the proof of (b), suppose $\Upsilon \colon \widehat{X} \to X$ is a connected covering space, $\hat{g} = \Upsilon^* g$, $p \in X$, $R > 0$, and $\{x_\nu\}$ is a sequence in $\widehat{X}$ such that $\operatorname{dist}_{\hat{g}}(x_\nu, p) < R$ for each ν. We must show that $\{x_\nu\}$ admits a convergent subsequence. Since $\{\Upsilon(x_\nu)\} \subset B_g(p; R) \Subset X$, we may assume that $\{\Upsilon(x_\nu)\}$ converges to some point $a \in X$. Fixing a contractible neighborhood U of a in X and a sufficiently small constant $\epsilon > 0$, we may also assume that $\{\Upsilon(x_\nu)\} \subset B_g(a; \epsilon) \Subset U$. Therefore, by replacing R with $R + \epsilon$ and each point x_ν with the unique point in $\Upsilon^{-1}(a)$ lying in the same connected component of $\Upsilon^{-1}(U)$ as x_ν, we may assume that $\Upsilon(x_\nu) = a$ for each ν. For each ν, there exists a piecewise C^∞ path γ_ν of length $< R$ from p to x_ν. Each of the homotopy classes $\left[\Upsilon(\gamma_1^{-1} * \gamma_\nu)\right]$ in $\pi_1(X, a)$ lies in the finite set $K(a, 2R)$, so the collection of terminal points $\{\, x_\nu \mid \nu = 1, 2, 3, \dots \}$ of the liftings $\left\{\gamma_1^{-1} * \gamma_\nu\right\}$ must be a finite set. The claim now follows. □

Lemma 1.3. *Let (X, g) be a connected reduced Hermitian complex space, let Y be a compact analytic subset of X, and let $\epsilon > 0$. Then there exists a constant $\delta > 0$ such that, for every connected covering space $\Upsilon \colon \widehat{X} \to X$ and for every pair of irreducible components A and B of $\widehat{Y} = \Upsilon^{-1}(Y)$, we have $\operatorname{dist}_{\hat{g}}(A \setminus N(A \cap B; \epsilon), B) > \delta$; where $\hat{g} = \Upsilon^* g$. In particular, for A and B disjoint, we have $\operatorname{dist}_{\hat{g}}(A, B) > \delta$.*

Remark. If $A \subset N(A \cap B; \epsilon)$, then we take $\operatorname{dist}_{\hat{g}}(A \setminus N(A \cap B; \epsilon), B) = \operatorname{dist}_{\hat{g}}(\emptyset, B) = \infty$.

Proof. We may choose open sets $\{V_j\}_{j=1}^m$ in X and contractible open sets $\{U_j\}_{j=1}^m$ in X such that $Y \subset \bigcup_{j=1}^m V_j$ and such that, for each $j = 1, \dots, m$, we have $V_j \Subset U_j$. We may fix a Lebesgue number $\eta > 0$ for the covering $\{V_j\}_{j=1}^m$ of Y with respect to the metric $\operatorname{dist}_g(\cdot, \cdot)$. In other words, for each point $p \in Y$, we have $B_g(p; \eta) \subset V_j$ for some j. We may also choose η so that $\eta < \epsilon$. We may then choose $\delta > 0$ so that $2\delta < \eta$ and so that, for each $j = 1, \dots, m$, we have $2\delta < \operatorname{dist}_g(\overline{V}_j \cap A \setminus N(A \cap B; \eta), B)$ for every pair of irreducible components A and B of $Y \cap U_j$.

Now suppose that $\Upsilon\colon \widehat{X} \to X$ is a connected covering space, $\hat{g} = \Upsilon^* g$, A and B are irreducible components of $\widehat{Y} = \Upsilon^{-1}(Y)$, and $\hat{a} \in A$ and $\hat{b} \in B$ with $\operatorname{dist}_{\hat{g}}(\hat{a}, \hat{b}) < 2\delta$. Setting $a = \Upsilon(\hat{a})$ and $b = \Upsilon(\hat{b})$, we then have $b \in B_g(a; 2\delta) \subset B_g(a; \eta) \subset V_j$ for some j. Hence

$$\hat{b} \in B_{\hat{g}}(\hat{a}; 2\delta) \subset B_{\hat{g}}(\hat{a}; \eta) \subset V \subset U,$$

where U is the connected component of $\Upsilon^{-1}(U_j)$ containing $\hat{a}$ (which Υ maps isomorphically onto U_j) and $V = \Upsilon^{-1}(V_j) \cap U$. We have $\hat{a} \in \widehat{A}_0$ and $\hat{b} \in \widehat{B}_0$ for some irreducible components $\widehat{A}_0$ of $A \cap U$ and $\widehat{B}_0$ of $B \cap U$ (and, therefore, of $\widehat{Y} \cap U$). Since Υ maps U isomorphically onto U_j, the sets $A_0 = \Upsilon(\widehat{A}_0)$ and $B_0 = \Upsilon(\widehat{B}_0)$ are irreducible components of $Y \cap U_j$. Since $\operatorname{dist}_g(a, b) < 2\delta$, the choice of δ implies that $a \in \overline{V}_j \cap A_0 \cap N(A_0 \cap B_0; \eta)$. Hence we may choose a piecewise C^∞ path λ of length less than η in X with $\lambda(0) = a$ and $\lambda(1) \in A_0 \cap B_0$. The lifting $\hat{\lambda}$ with $\hat{\lambda}(0) = \hat{a}$ lies entirely in U and, therefore, $\hat{\lambda}(1) \in \widehat{A}_0 \cap \widehat{B}_0 \subset A \cap B$. Hence $\operatorname{dist}_{\hat{g}}(\hat{a}, A \cap B) < \eta < \epsilon$ and it follows that $\operatorname{dist}_{\hat{g}}(A \setminus N(A \cap B; \epsilon), B) \geq 2\delta > \delta$. $\square$

2. Positive sums of eigenvalues of the Levi form

Throughout this section X will denote a reduced complex space, g will denote a Hermitian metric in X, and q will denote a positive integer.

Definition 2.1. Let φ be a real-valued function on an open subset Ω of X. We will say that φ is *of class* $\mathcal{W}^\infty(g, q)$ (*of class* $\mathcal{SP}^\infty(g, q)$) and write $\varphi \in \mathcal{W}^\infty(g, q)(\Omega)$ (respectively, $\varphi \in \mathcal{SP}^\infty(g, q)(\Omega)$) if, for every analytic subset Y of an open subset of Ω, every point $p \in Y$, every local holomorphic model (U, Φ, U') in Y with $p \in U$, and every Hermitian metric g' in U' with $\Phi^* g' = g \restriction_Y$ on U, there exists a C^∞ function φ' on a neighborhood V' of $\Phi(p)$ in U' such that $\varphi' \circ \Phi = \varphi$ on $V = \Phi^{-1}(V') \subset U$ and such that, for each point $z \in V'$, the trace of the restriction of the Levi-form $\mathcal{L}(\varphi')$ to any q-dimensional subspace of $\mathcal{T}^{(1)}_z V'$ with respect g' is nonnegative (respectively, positive); that is, for any g'-orthonormal collection of vectors $e_1, \dots, e_q \in \mathcal{T}^{(1)}_z V'$, we have

$$\sum_{i=1}^{q} \mathcal{L}(\varphi')(e_i, e_i) \geq 0 \quad \text{(respectively, } > 0).$$

Clearly, $\mathcal{W}^\infty(g, q) \subset \mathcal{W}^\infty(q)$ and $\mathcal{SP}^\infty(g, q) \subset \mathcal{SP}^\infty(q)$. For manifolds, these classes of functions were first introduced by Grauert and Riemenschneider [GR] and have been applied by others in many different contexts. The definition is stated in the above (rather cumbersome) form in order to guarantee the above inclusions as well as to guarantee invariance under sums (by independence of the local representation) and restrictions to

analytic subsets. One weakness in the definition of $\mathcal{W}^\infty(g,q)$ is that the local extension φ' is not assumed to be of class $\mathcal{W}^\infty(g',q)$; i.e. it is not assumed that φ' admits further extensions with Levi form satisfying the trace condition with respect to other local representations of g'. One could build the existence of such extensions into the definition (in fact, the functions produced in this paper will actually have such properties), but the above notion suffices for our purposes. For $\mathcal{SP}^\infty(g,q)$, the situation is much better. In fact, the following proposition provides an equivalent notion which is more easily checked (see, for example, [JNR] as well as [Ri], [De], [Co1], [J]):

Proposition 2.2. *A function φ on an open subset Ω of X is of class $\mathcal{SP}^\infty(g,q)$ if and only if, for every point $p\in\Omega$, there exist a local holomorphic model (U,Φ,U') in X with $p\in U\subset\Omega$ and a C^∞ function φ' on U' such that $\varphi'\circ\Phi=\varphi$ on U and such that, for each point $x\in U$, the trace of the restriction of the pullback of the Levi-form $\Phi^*\mathcal{L}(\varphi')$ to any q-dimensional subspace of $\mathcal{T}^{(1)}_x U$ with respect to g is positive.*

The analogous class in which the trace of the restriction of $\Phi^*\mathcal{L}(\varphi')$ is only assumed to be nonnegative is considered in [JNR]. The authors do not know whether or not this gives the same class $\mathcal{W}^\infty(g,q)$.

One also has the following global extension theorem of Richberg [Ri] (cf. Fritzsche [Fr], Demailly [De], Colţoiu [Co1], Joita [J], and [JNR]):

Theorem 2.3. *If Y is an analytic subset of X and $\varphi\in\mathcal{SP}^\infty(g\restriction_Y,q)(Y)$, then there exists a function ψ of class $\mathcal{SP}^\infty(g,q)$ on a neighborhood of Y in X such that $\psi=\varphi$ on Y.*

Combining this with modifications of the arguments of Greene and Wu [GW], Demailly [De], and Ohsawa [O], one gets the following (see [JNR]):

Theorem 2.4. *Let Y be a (properly embedded) analytic subset of X of dimension $\leq q$ with no compact irreducible components of dimension q. Then there exists a C^∞ exhaustion function φ on X which is of class $\mathcal{SP}^\infty(g,q)$ on a neighborhood of Y.*

It will also be convenient to have the following easy lemma:

Lemma 2.5. *Let χ be a C^∞ function on an interval (a,b) in $\mathbb{R}$ with $\chi'\geq 0$ and $\chi''\geq 0$ and let φ be a function of class $\mathcal{SP}^\infty(g,q)$ on an open set $\Omega\subset X$ with values in (a,b). Then the function $\psi\equiv\chi(\varphi)$ has the following properties:*

(a) *We have $\psi\in\mathcal{W}^\infty(g,q)(\Omega)$.*
(b) *If $\chi'>0$, then $\psi\in\mathcal{SP}^\infty(g,q)(\Omega)$.*

(c) *For any C^∞ function α with compact support in Ω, there is a constant $C = C(\varphi, \alpha) > 0$ such that $\psi + \alpha$ will be of class $\mathcal{SP}^\infty(g,q)$ on Ω for any choice of χ with $\chi' > 0$ (and $\chi'' \geq 0$) on (a,b) and $\chi' > C$ on $\varphi(\operatorname{supp}\alpha)$.*

Remark. Most of the $\mathcal{W}^\infty(g,q)$ functions to be constructed in this paper will locally be expressible as sums of functions of the form $\psi = \chi(\varphi)$ as in the above lemma.

3. Q-convexity in a covering of a neighborhood of a Stein manifold

In this section, we produce, in a covering, a function on a uniform neighborhood of the lifting of an embedded Stein manifold which is strongly q-convex near the lifting of a large compact set and equal to 0 elsewhere. More precisely, we prove the following:

Proposition 3.1. *Let (X,g) be a connected reduced Hermitian complex space of bounded local embedding dimension; let Y be a connected Stein manifold of dimension $q > 0$ which is properly embedded in X; and let Q, Ω_1, and Ω_2 be open subsets of X such that*

$$X \Supset \Omega_1 \Supset \Omega_2 \Supset Q;$$

$Q \cap Y$ and $\Omega_j \cap Y$ for $j = 1,2$ are nonempty, connected, and relatively compact in Y; $\overline{Q \cap Y} = \overline{Q} \cap Y$; and $\overline{\Omega_j \cap Y} = \overline{\Omega}_j \cap Y$ for $j = 1,2$. Then there exists a connected neighborhood Θ_1 of Y in X and a nonnegative C^∞ function α on Θ_1 satisfying the following.

(a) *We have*
  (i) *On $\Theta_1 \setminus \Omega_1$, $\alpha \equiv 0$;*
  (ii) *On $\Theta_1 \cap \Omega_2$, $\alpha > 0$;*
  (iii) *On some neighborhood of $\Theta_1 \setminus Q$, α is of class $\mathcal{W}^\infty(g,q)$; and*
  (iv) *For $B = \{\, x \in \Theta_1 \mid \alpha(x) > 0 \,\} \supset \Theta_1 \cap \Omega_2$, α is of class $\mathcal{SP}^\infty(g,q)$ on a neighborhood of $B \setminus Q$.*

(b) *Suppose Ω_3 and Ω_4 are open subsets of X such that*

$$\Omega_2 \Supset \Omega_3 \Supset \Omega_4 \Supset Q,$$

and, for $j = 3,4$, $\Omega_j \cap Y$ is connected and $\overline{\Omega_j \cap Y} = \overline{\Omega}_j \cap Y$. Suppose also that $\{\omega_\nu\}_{\nu \in N}$ is a family of real-valued C^∞ functions with compact support in B. Then, for some connected neighborhood Θ_2 of Y in Θ_1, for any family of sufficiently large positive constants $\{R_\nu\}_{\nu \in N}$, for every connected infinite covering space $\Upsilon : \widehat{X} \to X$ in which $\widehat{Y} = \Upsilon^{-1}(Y)$ and $\Upsilon^{-1}(\Omega_4 \cap Y)$ are connected, and for every positive continuous function θ on $\widehat{X}$, there is a nonnegative C^∞ function α_0

on $\widehat{\Theta}_2 = \Upsilon^{-1}(\Theta_2)$ such that, if $\widehat{\Omega}_j = \Upsilon^{-1}(\Omega_j)$ for $j = 1, 2, 3, 4$, $\hat{\alpha} = \alpha \circ \Upsilon$, and $\widehat{B} = \Upsilon^{-1}(B)$, then

(i) *On $\widehat{\Theta}_2$, $\alpha_0 \geq \hat{\alpha}$;*
(ii) *On $\widehat{\Theta}_2 \setminus \widehat{\Omega}_3$, $\alpha_0 = \hat{\alpha}$;*
(iii) *For each point $\nu \in N$, the function $R_\nu \cdot \alpha_0 + \omega_\nu \circ \Upsilon$ will be C^∞ strongly q-convex on $\widehat{B} \cap \widehat{\Theta}_2$; and*
(iv) *On $\widehat{\Theta}_2 \cap \widehat{\Omega}_4$, $\alpha_0 > \theta$.*

Lemma 3.2. *Let (X, g) be a Hermitian complex manifold, let Z be a (properly embedded) complex submanifold, and let α be a real-valued C^∞ function on X such that $\alpha \restriction_Z$ is of class $\mathcal{SP}^\infty(g \restriction_Z, q)$ on Z. Suppose ρ is a C^∞ function on X such that, for each point $p \in Z$, we have $\rho(p) = 0$, $(d\rho)_p = 0$, and $\mathcal{L}(\rho)(v, v) > 0$ for each tangent vector $v \in T^{1,0}_p X \setminus T^{1,0} Z$ (in particular, $\mathcal{L}(\rho)(v, v) \geq 0$ for each $v \in T^{1,0}_p X$). Then there exists a positive continuous function λ_0 on Z such that, for every function $\lambda \in C^\infty(X)$ with $\lambda > \lambda_0$ on Z, the function $\beta \equiv \alpha + \lambda \cdot \rho$ is of class $\mathcal{SP}^\infty(g, q)$ on a neighborhood of Z in X (depending on the choice of λ).*

Proof. Suppose λ is a positive C^∞ function on X and $\beta \equiv \alpha + \lambda \cdot \rho$. For each point $p \in Z$ and each tangent vector $v \in T^{1,0}_p X$, we have

$$\mathcal{L}(\beta)(v, v) = \mathcal{L}(\alpha)(v, v) + \lambda(p) \cdot \mathcal{L}(\rho)(v, v) \geq \mathcal{L}(\alpha)(v, v).$$

Let E_X and E_Z be the set of orthonormal q-frames in $\mathcal{T}^{(q)} X$ and $\mathcal{T}^{(q)} Z$, respectively. By the definition of $\mathcal{SP}^\infty(g, q)$ and continuity, there is a neighborhood W of E_Z in $\mathcal{T}^{(q)} X$ such that

$$\sum_{j=1}^{q} \mathcal{L}(\alpha)(v_j, v_j) > 0 \qquad \forall\, v = (v_1, \ldots, v_q) \in W.$$

On the other hand, for each $(e_1, \ldots, e_q) \in (E_X \setminus W) \cap [\Pi_{\mathcal{T}^{(q)} X}]^{-1}(Z)$, we have

$$\sum_{j=1}^{q} \mathcal{L}(\rho)(e_j, e_j) > 0.$$

Therefore, if λ grows sufficiently quickly at infinity on Z, then we will have

$$\sum_{j=1}^{q} \mathcal{L}(\beta)(e_j, e_j) > 0 \qquad \forall\, (e_1, \ldots, e_q) \in E_X \cap [\Pi_{\mathcal{T}^{(q)} X}]^{-1}(Z),$$

and it follows that β will be of class $\mathcal{SP}^\infty(g, q)$ on a neighborhood of Z. $\square$

Lemma 3.3. *Let X be a Stein manifold and let Z be a (properly embedded) complex submanifold. Then there exists a nonnegative C^∞ plurisubharmonic function ρ on X such*

that $\rho(p) = 0$ and $(d\rho)_p = 0$ for each point $p \in Z$ and $\mathcal{L}(\rho)(v, v) > 0$ for each nonzero tangent vector $v \in T^{1,0}X \setminus T^{1,0}Z$.

Proof. For each point $p \in X$, Cartan's Theorem A provides holomorphic functions

$$f_p^{(1)}, \dots, f_p^{(m_p)} \in H^0(X, \mathcal{I}_{\{p\}\cup Z})$$

generating the ideal sheaf $\mathcal{I}_{\{p\}\cup Z}$ at each point in a neighborhood U_p of p in X. Setting $f_p = \left(f_p^{(1)}, \dots, f_p^{(m_p)}\right)$, we see that, for any nonzero vector $v \in T^{1,0}U_p$, we have $df_p(v) = 0$ if and only if $v \in T^{1,0}Z$. Forming a countable cover $\{U_{p_\nu}\}$ of X by such sets and choosing a sequence of positive numbers $\{\epsilon_\nu\}$ converging to 0 sufficiently fast, we get the function $\rho \equiv \sum \epsilon_\nu |f_{p_\nu}|^2$ with the required properties. □

As in [Fra], we will also apply the following theorem of Demailly [De] (see Theorem 1.13 of [NR]):

Theorem 3.4. *Let D be a connected closed noncompact subset of a Hermitian manifold (X, g), let U be a connected neighborhood of D in X, and let θ be a positive continuous function on X. Then there exists a C^∞ subharmonic function α on X such that $\alpha \equiv 0$ on $X \setminus U$, $\alpha > 0$ and $\Delta\alpha > 0$ on U, and $\alpha > \theta$ and $\Delta\alpha > \theta$ on D.*

Proof of Proposition 3.1. We first prove the proposition for X a connected open subset of $\mathbb{C}^n$ and Y closed in $\mathbb{C}^n$. Let $\pi\colon \mathcal{N} = [T^{1,0}\mathbb{C}^n \restriction_Y]/T^{1,0}Y \to Y$ be the normal bundle with the Hermitian metric h induced by the Euclidean metric. Then there exists a positive continuous function σ on Y and a biholomorphism $\Psi\colon N \to M$ of the neighborhood

$$N = \{\, \xi \in \mathcal{N} \mid |\xi|_h < \sigma(\pi(\xi)) \,\}$$

of the 0-section onto a connected neighborhood M of Y in X such that $\Psi(0_y) = y$ for each point $y \in Y$ (see, for example, [FoR]). Setting $\Lambda(t, x) = \Lambda_t(x) = \Psi(t\Psi^{-1}(x))$ for each $(t, x) \in [0, 1] \times M$, we get a C^∞ strong deformation retraction $\Lambda\colon [0, 1] \times M \to M$ of M onto Y for which the map $\Lambda_t\colon M \to M$ is holomorphic for each $t \in [0, 1]$. In particular, $\Lambda_0\colon M \to Y$ is a holomorphic submersion which is equal to the identity on Y.

According to Lemma 3.3, there exists a nonnegative C^∞ plurisubharmonic function ρ on $\mathbb{C}^n$ such that $\rho(z) = 0$ and $(d\rho)_z = 0$ for each point $z \in Y$ and $\mathcal{L}(\rho)(v, v) > 0$ for each nonzero tangent vector $v \in T^{1,0}\mathbb{C}^n \setminus T^{1,0}Y$.

Fixing a point $p \in Q \cap Y$, applying Theorem 3.4 in the Hermitian manifold $Y \setminus \{p\}$ to the connected relatively open set $\Omega_1 \cap Y \setminus \{p\}$ and the noncompact connected relatively closed set $\overline{\Omega}_2 \cap Y \setminus \{p\} \subset \Omega_1 \cap Y \setminus \{p\}$, and patching with a positive C^∞ function near p in Q, we get a nonnegative C^∞ function β on Y such that

(3.1.1) On $Y \setminus \Omega_1$, $\beta \equiv 0$;
(3.1.2) On $\Omega_1 \cap Y$, $\beta > 0$; and
(3.1.3) On $\Omega_1 \cap Y \setminus Q$, $\Delta_g \beta > 0$.

We may now choose a constant δ with $0 < 3\delta < \min_{\overline{\Omega}_2 \cap Y} \beta$ and open sets U_1 and Q_0 in X such that

$$\Omega_1 \Supset U_1 \Supset \Omega_2, \quad Q \Supset Q_0, \quad \beta < \delta \text{ on } Y \setminus U_1, \text{ and } \Delta_g \beta > 0 \text{ on } \Omega_1 \cap Y \setminus Q_0.$$

According to Lemma 3.2, if we fix a positive function $\lambda \in C^\infty(\Omega_1)$ which is constant on U_1 and sufficiently large on $\Omega_1 \cap Y$, then the function $\beta \circ \Lambda_0 + \lambda\rho$ will be of class $\mathcal{SP}^\infty(g, q)$ on a neighborhood of $U_2 \setminus Q_0$ for some neighborhood U_2 of $\Omega_1 \cap Y$ in $\Omega_1 \cap M$. Furthermore, choosing U_2 sufficiently small, we get $\beta \circ \Lambda_0 + \lambda\rho < \delta$ ($> 3\delta$) on $U_2 \setminus U_1$ (respectively, on $U_2 \cap \Omega_2$). Therefore, choosing an open set U_3 in X with $Y \setminus \Omega_1 \subset U_3 \subset \overline{U}_3 \subset M \setminus \overline{U}_1$; a C^∞ function $\chi \colon \mathbb{R} \to [0, \infty)$ such that $\chi' \geq 0$ and $\chi'' \geq 0$ on $\mathbb{R}$, $\chi(t) \equiv 0$ for $t \leq \delta$, and $\chi(t) = t - 2\delta$ for $t \geq 3\delta$; and a connected neighborhood Θ_1 of Y in $U_3 \cup U_2$, we may define a C^∞ function α on Θ_1 by setting $\alpha = 0$ on $\Theta_1 \setminus U_2$ and $\alpha = \chi(\beta \circ \Lambda_0 + \lambda\rho)$ on $\Theta_1 \cap U_2$. It is now easy to verify that α has the properties described in (a) (using Lemma 2.5 to get (iii)).

For the construction of the neighborhood Θ_2 as in (b), we choose open sets V_1, V_2, and V_3 in X such that $\Omega_3 \Supset V_1 \Supset V_2 \Supset V_3 \Supset \Omega_4$ and such that, for $j = 1, 2, 3$, $V_j \cap Y$ is connected and $\overline{V_j \cap Y} = \overline{V}_j \cap Y$. If $\sigma_0 < \sigma$ is a sufficiently small positive continuous function Y, then the connected open set $\Theta_2 \equiv \Psi(\{\, \xi \in \mathcal{N} \mid |\xi|_h < \sigma_0(\pi(\xi)) \,\})$, will satisfy

$$Y \subset \Theta_2 \subset \overline{\Theta}_2 \subset \Theta_1, \quad \Lambda_0^{-1}(V_2 \cap Y) \cap \Theta_2 \subset V_1, \text{ and } \Lambda_0(\Omega_4 \cap \Theta_2) \subset V_3 \cap Y.$$

We may also choose a nonnegative C^∞ function γ on $\mathbb{C}^n$ such that $\operatorname{supp}\gamma \subset \Omega_3 \cap \Theta_1$, γ is C^∞ strictly plurisubharmonic on a neighborhood of $\overline{V}_1 \cap \overline{\Theta}_2$, and $\alpha + \gamma$ is of class $\mathcal{SP}^\infty(g, q)$ on a neighborhood of $B \cap \overline{\Theta}_2 \setminus Q$; where $B = \{\, x \in \Theta_1 \mid \alpha(x) > 0 \,\} \supset \Theta_1 \cap \Omega_2$.

Suppose now that $\Upsilon : \widehat{X} \to X$ is a connected infinite covering space in which $\widehat{Y} = \Upsilon^{-1}(Y)$ and $\Upsilon^{-1}(\Omega_4 \cap Y)$ are connected. Let θ be a positive continuous function on $\widehat{X}$, let $\widehat{M} = \Upsilon^{-1}(M)$, let $\widehat{Q} = \Upsilon^{-1}(Q)$, let $\widehat{\Theta}_j = \Upsilon^{-1}(\Theta_j)$ for $j = 1, 2$, let $\widehat{\Omega}_j = \Upsilon^{-1}(\Omega_j)$ for $j = 1, 2, 3, 4$, let $\widehat{V}_j = \Upsilon^{-1}(V_j)$ for $j = 1, 2, 3$, let $\hat{\alpha} = \alpha \circ \Upsilon$, let $\hat{\beta} = \beta \circ \Upsilon \restriction_{\widehat{Y}}$, let $\hat{\rho} = \rho \circ \Upsilon$, let $\hat{\lambda} = \lambda \circ \Upsilon$, and let $\hat{g} = \Upsilon^* g$. Observe that the map Λ lifts to a unique C^∞ strong deformation retraction $\hat{\Lambda}$ of $\widehat{M}$ onto $\widehat{Y}$. The restriction of the corresponding holomorphic submersion $\hat{\Lambda}_0 \colon \widehat{M} \to \widehat{Y}$ to $\overline{\widehat{\Theta}_2}$ is a proper map and hence we may choose a positive continuous function θ_0 on $\widehat{Y}$ such that $\theta_0 \circ \hat{\Lambda} > \theta$ on $\widehat{\Theta}_2$.

Applying Theorem 3.4 in the Hermitian manifold $(\widehat{Y}, \hat{g} \restriction_{\widehat{Y}})$ to the connected relatively open set $\widehat{V}_2 \cap \widehat{Y}$ and the noncompact connected closed set $\overline{\widehat{V}_3} \cap \widehat{Y}$, we get a nonnegative C^∞ function β_0 on $\widehat{Y}$ such that

(3.1.4) On $\widehat{Y} \setminus \widehat{V}_2$, $\beta_0 \equiv 0$;
(3.1.5) On $\widehat{V}_2 \cap \widehat{Y}$, $\beta_0 > 0$ and $\Delta_{\hat{g}}\beta_0 > 0$; and
(3.1.6) On $\widehat{V}_3 \cap \widehat{Y}$, $\beta_0 > \theta_0$ and $\Delta_{\hat{g}}(\hat{\beta} + \beta_0) > 0$.

In particular, $\Delta_{\hat{g}}(\hat{\beta} + \beta_0) > 0$ on $\widehat{\Omega}_1 \cap \widehat{Y}$. Setting $\widehat{B} = \Upsilon^{-1}(B)$, $\hat{\gamma} = \gamma \circ \Upsilon$, and $\alpha_1 = (\hat{\alpha} + \beta_0 \circ \hat{\Lambda}_0) \restriction_{\widehat{\Theta}_2}$, we will show that the C^∞ function $\alpha_0 = \alpha_1 + \hat{\gamma}$ has the required properties.

Clearly, we have $\alpha_0 \geq \hat{\alpha}$ on $\widehat{\Theta}_2$. Since $\alpha_1 = \hat{\alpha}$ on $\widehat{\Theta}_2 \setminus \hat{\Lambda}_0^{-1}(\widehat{V}_2 \cap \widehat{Y}) \supset \widehat{\Theta}_2 \setminus \widehat{V}_1 \supset \widehat{\Theta}_2 \setminus \widehat{\Omega}_3$ and $\operatorname{supp} \gamma \subset \Omega_3$, we have $\alpha_0 = \hat{\alpha}$ on $\widehat{\Theta}_2 \setminus \widehat{\Omega}_3$. On the set $\hat{\Lambda}_0^{-1}(\widehat{V}_3 \cap \widehat{Y}) \cap \widehat{\Theta}_2 \supset \widehat{\Omega}_4 \cap \widehat{\Theta}_2$, we have $\alpha_0 \geq \beta_0 \circ \hat{\Lambda}_0 > \theta_0 \circ \Lambda_0 > \theta$.

It remains to verify the condition (iii) in part (b). Since $\alpha_0 = \hat{\alpha} + \hat{\gamma}$ on $\widehat{\Theta}_2 \setminus \hat{\Lambda}_0^{-1}(\widehat{V}_2 \cap \widehat{Y}) \subset \widehat{\Theta}_2 \setminus \widehat{Q}$, since $\alpha + \gamma$ is of class $\mathcal{SP}^\infty(g, q)$ on a neighborhood of $B \cap \overline{\Theta}_2 \setminus Q$, and since γ is C^∞ strictly plurisubharmonic on a neighborhood of $\overline{V}_1 \cap \overline{\Theta}_2$, we need only show that α_1 is C^∞ strongly q-convex in a neighborhood of each point $x \in \widehat{\Theta}_2 \cap \hat{\Lambda}_0^{-1}\left(\overline{\widehat{V}_2} \cap \widehat{Y}\right)$. For this, observe that, since $y = \hat{\Lambda}_0(x) \in \widehat{\Omega}_1 \cap \widehat{Y}$, we have $\Delta_{\hat{g}}(\hat{\beta} + \beta_0)(y) > 0$. Hence there exists a 1-dimensional vector subspace $\mathcal{V}$ of $T_y^{1,0}\widehat{Y}$ such that $\mathcal{L}\left(\hat{\beta} + \beta_0\right) > 0$ on $\mathcal{V} \setminus \{0\}$. Since Λ_0 is a submersion, the inverse image $\mathcal{U} = [(\Lambda_0)_*]_x^{-1}(\mathcal{V})$ is a vector subspace of dimension $n - q + 1$ in $T_y^{1,0}\widehat{X}$. By construction, on some neighborhood of x we have $\hat{\beta} \circ \hat{\Lambda}_0 + \hat{\lambda}\hat{\rho} > 3\delta$ and hence

$$\alpha_1 = \chi(\hat{\beta} \circ \hat{\Lambda}_0 + \hat{\lambda}\hat{\rho}) + \beta_0 \circ \hat{\Lambda}_0 = \hat{\beta} \circ \hat{\Lambda}_0 + \hat{\lambda}\hat{\rho} - 2\delta + \beta_0 \circ \hat{\Lambda}_0 = (\hat{\beta} + \beta_0) \circ \hat{\Lambda}_0 + \hat{\lambda}\hat{\rho} - 2\delta.$$

Moreover, $\hat{\lambda}$ is constant near x. Thus for each tangent vector $v \in \mathcal{U} \setminus \{0\}$, we have

$$\mathcal{L}(\alpha_1)(v, v) = \mathcal{L}\left(\hat{\beta} + \beta_0\right)\left((\hat{\Lambda}_0)_* v, (\hat{\Lambda}_0)_* v\right) + \hat{\lambda}(x)\mathcal{L}(\hat{\rho})(v, v).$$

Both terms on the right-hand side are nonnegative, the first is positive if $(\hat{\Lambda}_0)_* v \neq 0$, and the second is positive if $(\hat{\Lambda}_0)_* v = 0$. Thus α_1 is C^∞ strongly q-convex near x.

We now consider the case of a general connected reduced complex space X of bounded local holomorphic embedding dimension. According to the Stein neighborhood theorem of Siu [Si] and the embedding theorem of Remmert [R], Narasimhan [Ns], and Bishop [B], some connected neighborhood of Y in X admits a proper holomorphic embedding into $\mathbb{C}^n$ for some positive integer n. Thus there exists a proper holomorphic embedding $\Phi \colon Z \to X'$ of a connected neighborhood Z of Y in X into a connected open subset X' of $\mathbb{C}^n$ such

that $Y' = \Phi(Y)$ is closed in $\mathbb{C}^n$, Y is a continuous strong deformation retract of Z, and $Z' = \Phi(Z)$ is a continuous strong deformation retract of X'. We may also fix a Hermitian metric g' on X' such that $\Phi^* g' = g$ on Z. Finally, we may fix a connected neighborhood P_1 of Y' in $\mathbb{C}^n$ with $\overline{P}_1 \subset X'$.

Given open sets Ω_1, Ω_2, and Q as in the statement of the proposition, we may choose open sets Ω'_1, Ω'_2, and Q' in $\mathbb{C}^n$ such that $X' \Supset \Omega'_1 \Supset \Omega'_2 \Supset Q'$, $Q' \cap Z' \cap P_1 = \Phi(Q \cap Z) \cap P_1$, $\Omega'_j \cap Z' \cap P_1 = \Phi(\Omega_j \cap Z) \cap P_1$ for $j = 1, 2$, $\overline{Q' \cap Z'} = \overline{Q'} \cap Z'$, and $\overline{\Omega'_j \cap Z'} = \overline{\Omega'_j} \cap Z'$ for $j = 1, 2$. By the above, there exists a neighborhood $\Theta'_1 \subset P_1$ of Y' and a C^∞ function α' satisfying the conditions in part (a) relative to these choices of sets in $\mathbb{C}^n$. The connected component Θ_1 of $\Phi^{-1}(\Theta'_1)$ containing Y and the function $\alpha = \alpha' \circ \Phi \restriction_{\Theta_1}$ on Θ_1 then satisfy the conditions in part (a) in X.

Fix a connected open set P_2 in $\mathbb{C}^n$ with $\overline{P}_2 \subset \Theta_1$. Given open sets Ω_3 and Ω_4 as in part (b), we may choose open sets Ω'_3 and Ω'_4 in $\mathbb{C}^n$ such that $\Omega'_2 \Supset \Omega'_3 \Supset \Omega'_4 \Supset Q'$, $\Omega'_j \cap Z' \cap P_2 = \Phi(\Omega_j \cap Z) \cap P_2$ for $j = 3, 4$, and $\overline{\Omega'_j \cap Z'} = \overline{\Omega'_j} \cap Z'$ for $j = 3, 4$. Set $B' = \{\, x \in \Theta'_1 \mid \alpha'(x) > 0 \,\}$ and $B = \Phi^{-1}(B') \cap \Theta_1 = \{\, x \in \Theta_1 \mid \alpha(x) > 0 \,\}$. Given a family of functions $\{\omega_\nu\}_{\nu \in N}$ as in the statement of part (b), we may form a family of real-valued C^∞ functions $\{\omega'_\nu\}_{\nu \in N}$ with compact support in B' such that $\omega'_\nu \circ \Phi = \omega_\nu$ on B for each ν.

Again, for some neighborhood $\Theta'_2 \subset P_2$ of Y' and for sufficiently large positive constants $\{R_\nu\}$, we get the conditions in part (b) relative to the above choices and any covering space of X' (with the required properties). Let Θ_2 be the connected component of $\Phi^{-1}(\Theta'_2)$ containing Y. For any infinite covering space $\Upsilon \colon \widehat{X} \to X$ in which $\widehat{Y} = \Upsilon^{-1}(Y)$ and $\Upsilon^{-1}(\Omega_4 \cap Y)$ are connected and for any positive continuous function θ on $\widehat{X}$, we get a corresponding covering $\Upsilon' \colon \widehat{X}' \to X'$ such that Φ lifts to a proper holomorphic embedding $\hat{\Phi}$ of $\widehat{Z} = \Upsilon^{-1}(Z)$ into $\widehat{X}'$ and we may choose a positive continuous function θ' on $\widehat{X}'$ with $\theta' \circ \hat{\Phi} = \theta$ on $\widehat{Z}$. Forming the corresponding function α'_0 on $\widehat{\Theta}'_2 = (\Upsilon')^{-1}(\Theta'_2)$, we get the desired function $\alpha_0 = \alpha'_0 \circ \hat{\Phi} \restriction_{\widehat{\Theta}_2}$ on $\widehat{\Theta}_2 = \Upsilon^{-1}(\Theta_2)$. $\square$

4. A uniformly quasi-plurisubharmonic function

We recall that an upper semi-continuous function $\varphi \colon M \to [-\infty, \infty)$ on a complex manifold M is called *plurisubharmonic* if, for every holomorphic mapping $f \colon D \to M$ of a disk $D \subset \mathbb{C}$ into M, the function $\psi = \varphi \circ f$ is subharmonic in D; that is, for every continuous function u on a compact set $K \subset D$ which is harmonic on the interior $\overset{\circ}{K}$ and which satisfies $u \geq \psi$ on ∂K, we have $u \geq \psi$ on K. The function φ is *strictly plurisubharmonic* if, for every real-valued C^∞ function ρ with compact support in M, the function $\varphi + \epsilon\rho$

is plurisubharmonic for every sufficiently small $\epsilon > 0$. Let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$. A function $\varphi\colon X \to [-\infty, \infty)$ on a reduced complex space X is *plurisubharmonic* (*strictly plurisubharmonic, C^k plurisubharmonic, C^k strictly plurisubharmonic*) if, for each point in $p \in X$, there is a local holomorphic chart (U, Φ, U') with $p \in U$ and a function φ' on U' such that φ' is plurisubharmonic (respectively, strictly plurisubharmonic, plurisubharmonic and of class C^k, strictly plurisubharmonic and of class C^k) and such that $\varphi = \varphi' \circ \Phi$ on U. The function φ is *quasi-plurisubharmonic* (*C^k quasi-plurisubharmonic*) if, locally, φ is the sum of a real analytic function and a plurisubharmonic (respectively, C^k plurisubharmonic) function. Clearly, for $k \in \mathbb{Z}_{\geq 2} \cup \{\infty, \omega\}$, a C^k function is C^k quasi-plurisubharmonic. By a theorem of Richberg [Ri], a continuous function which is strictly plurisubharmonic is automatically C^0 strictly plurisubharmonic. However, the corresponding property for C^k functions does not always hold on a complex space (see, for example, Smith [Sm]). Finally, we observe that the class of C^∞ strongly 1-convex functions is precisely the class of C^∞ strictly plurisubharmonic functions and $\mathcal{W}^\infty(1)$ is precisely the class of C^∞ plurisubharmonic functions (see the introduction).

The goal of this section is to produce a function on each covering space which is, in a sense, uniformly quasi-plurisubharmonic and which has a logarithmic singularity along the lifting of a given compact analytic set (cf. Lemma 5 of Demailly [De]). For this, it will be convenient to have the following terminology:

Definition 4.1. Let A be a (properly embedded) analytic subset of a reduced complex space X and let $\varphi\colon \Omega \to [-\infty, \infty)$ be a mapping on an open subset Ω of X. We will say that φ is of class $\mathcal{W}_A^\infty$ ($\mathcal{SP}_A^\infty$, $\mathcal{Q}_A^\infty$) if, for each point $p \in \Omega$, there is a local holomorphic model (U, Φ, U') in X with $p \in U \subset \Omega$ and a plurisubharmonic (respectively, strictly plurisubharmonic, quasi-plurisubharmonic) function $\varphi'\colon U' \to [-\infty, \infty)$ such that $\varphi = \varphi' \circ \Phi$ on U, φ' is real-valued and C^∞ on $U' \setminus \Phi(A \cap U)$, φ' is continuous on U' as a mapping into $[-\infty, \infty)$, and $\varphi' = -\infty$ on $\Phi(A \cap U)$.

The main goal of this section is the following:

Proposition 4.2. *Let (X, g) be a connected reduced Hermitian complex space, let Y be a compact analytic subset of X, let Ω be a relatively compact neighborhood of Y in X, let $\{D_\nu\}_{\nu=1}^l$ be relatively compact open subsets of X, and let ρ_ν be a C^∞ strictly plurisubharmonic function on a neighborhood of $\overline{D}_\nu$ for each $\nu = 1, \dots, l$. Then, for some choice of nondecreasing continuous maps*

$$\eta_-\colon [0, \infty] \to [-\infty, 0] \qquad \textit{and} \qquad \eta_+\colon [0, \infty] \to [-\infty, 0]$$

with $\eta_- \leq \eta_+$, $\eta_+(0) = -\infty$, and $\eta_- > -\infty$ on $(0, \infty]$; for every sufficiently large $R > 0$; for every connected covering space $\Upsilon \colon \widehat{X} \to X$; and for every analytic set Z equal to a union of irreducible components of $\widehat{Y} \equiv \Upsilon^{-1}(Y)$; there exists a continuous function $\alpha \colon \widehat{X} \to [-\infty, 0]$ with the following properties:

(i) *The function α is of class $\mathcal{Q}^{\infty}_{Z}$ on $\widehat{X}$;*

(ii) *We have* $\operatorname{supp} \alpha \subset \widehat{\Omega} \equiv \Upsilon^{-1}(\Omega)$;

(iii) *For each $\nu = 1, \dots, l$, $\alpha + R \cdot \rho_\nu \circ \Upsilon$ is of class $\mathcal{SP}^{\infty}_{Z}$ on $\Upsilon^{-1}(D_\nu)$;*

(iv) *For $\hat{g} = \Upsilon^* g$, we have*

$$\eta_-(\operatorname{dist}_{\hat{g}}(x, Z)) \leq \alpha(x) \leq \eta_+(\operatorname{dist}_{\hat{g}}(x, Z)) \qquad \forall\, x \in \widehat{X}.$$

Remarks. 1. For $Z = \emptyset$, we have $\operatorname{dist}_{\hat{g}}(\cdot, Z) \equiv \infty$.

2. Clearly, we may choose $R = 1$ (by replacing α and $\eta_\pm$ with $R^{-1}\alpha$ and $R^{-1}\eta_\pm$, respectively), but it will be more convenient for the proof to allow R to vary.

3. The condition $\mathcal{SP}^{\infty}_{Z}$ is slightly stronger than necessary for our purposes, but working with this class allows one to write some of the arguments more efficiently.

The proof of Proposition 4.2 is a modification of Demailly's construction of a quasi-plurisubharmonic function with logarithmic singularities along a given analytic subset (Lemma 5 of [De]). The function is obtained by patching local quasi-plurisubharmonic functions using the following C^∞ version of the maximum function for which the given properties are easy to check:

Lemma 4.3. *Let $\kappa \colon \mathbb{R} \to [0, \infty)$ be a C^∞ function such that* $\operatorname{supp} \kappa \subset (-1, 1)$, $\int_{\mathbb{R}} \kappa(u)\, du = 1$, *and* $\int_{\mathbb{R}} u\kappa(u)\, du = 0$. *For each $d \in \mathbb{Z}_{>0}$, let $\mathcal{M}_d \colon \mathbb{R}^d \to \mathbb{R}$ be the function given by*

$$\mathcal{M}_d(t) = \int_{\mathbb{R}^d} \left[\max_{1 \leq j \leq d} (t_j + u_j) \right] \prod_{1 \leq j \leq d} \kappa(u_j)\, du_j$$

for each $t = (t_1, \dots, t_d) \in \mathbb{R}^d$. Then

(a) *For each $t = (t_1, \dots, t_d) \in \mathbb{R}^d$,*

$$\mathcal{M}_d(t) = \int_{\mathbb{R}^m} \left[\max_{1 \leq j \leq d} u_j \right] \prod_{1 \leq j \leq d} \kappa(u_j - t_j)\, du_j.$$

(b) *$\mathcal{M}_d$ is C^∞, convex, and symmetric in $t_1, \dots, t_d$.*

(c) *$\mathcal{M}_d(t_1, \dots, t_d)$ is nondecreasing in each variable t_j.*

(d) *For every $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ and $s \in \mathbb{R}$, we have*

$$\mathcal{M}_d(t_1 + s, \dots, t_d + s) = \mathcal{M}_d(t) + s.$$

(e) *For every* $t \in \mathbb{R}^d$, $\max(t) \leq \mathcal{M}_d(t) \leq \max(t) + 1$.

(f) *If* $t' = (t_0, t_1, \dots, t_d) = (t_0, t) \in \mathbb{R}^{d+1}$ *with* $t_0 \leq t_1 - 2$, *then* $\mathcal{M}_{d+1}(t') = \mathcal{M}_d(t)$.

(g) *For* $d = 1$, $\mathcal{M}_d(t) = t$ *for each* $t \in \mathbb{R}$.

(h) *For each* $j = 1, \dots, d$, *we have* $0 \leq \frac{\partial}{\partial t_j}\mathcal{M}_d(t_1, \dots, t_d) \leq 1$.

(i) *If* $\varphi = (\varphi_1, \dots, \varphi_d)$ *is a* d*-tuple of* C^∞ *real-valued functions on a complex manifold* X, *then*

$$\mathcal{L}(\mathcal{M}_d(\varphi))(v, v) \geq \min_{1 \leq j \leq d} \mathcal{L}(\varphi_j)(v, v) \qquad \forall\, v \in T^{1,0}X.$$

The following is Demailly's construction:

Lemma 4.4 (See Lemma 5 of [De]). *Let* X *be a reduced complex space, let* A *be an analytic subset of* X, *let* $\{U_j\}_{j \in J}$ *and* $\{V_j\}_{j \in J}$ *be locally finite coverings of* X *by open sets with* $\overline{V}_j \subset U_j$ *for each* $j \in J$, *and, for each index* $j \in J$, *let* λ_j *be a* C^∞ *function on* V_j *with* $\lambda_j \to -\infty$ *at* ∂V_j *and let*

$$F_j = \left(f_j^{(1)}, \dots, f_j^{(N_j)}\right) : U_j \to \mathbb{C}^{N_j}$$

be a holomorphic map such that the coordinate functions generate the ideal sheaf $\mathcal{I}_{(A \cap U_j)}$ *at each point in* U_j. *Let* $\alpha\colon X \to [-\infty, \infty)$ *be the mapping defined by setting* $\alpha \equiv -\infty$ *on* A *and, for* $x \in X \setminus A$, *setting*

$$\alpha(x) = \mathcal{M}_d\left[\left(\log|F_{j_1}(x)|^2 + \lambda_{j_1}(x)\right), \dots, \left(\log|F_{j_d}(x)|^2 + \lambda_{j_d}(x)\right)\right];$$

where $j_1, \dots, j_d$ *are the distinct indices with* $x \in V_j$ *if and only if* $j \in \{j_1, \dots, j_d\}$. *Then* α *is of class* $\mathcal{Q}_A^\infty$ *on* X.

Proof. Given a point $x \in X$, we have distinct indices $j_1, \dots, j_d, j_{d+1}, \dots, j_m \in J$ such that $x \in V_j$ (respectively, $x \in \overline{V}_j$) if and only if $j \in \{j_1, \dots, j_d\}$ (respectively, $j \in \{j_1, \dots, j_m\}$). Thus we may choose a neighborhood W of x with

$$W \Subset V_{j_1} \cap \cdots \cap V_{j_d} \cap U_{j_{d+1}} \cap \cdots \cap U_{j_m} \text{ and } W \cap V_j = \emptyset \quad \forall\, j \in J \setminus \{j_1, \dots, j_m\}.$$

Clearly, the functions $|F_{j_\mu}|/|F_{j_\nu}|$ are bounded on $\overline{W}$ for $\mu, \nu = 1, \dots, m$. On the other hand, λ_{j_ν} is bounded on $\overline{W}$ for $\nu = 1, \dots, d$ and $\lambda_{j_\nu} \to -\infty$ at $x \in \partial V_{j_\nu}$ for $\nu = d+1, \dots, m$. Hence, choosing W sufficiently small, we get

$$\log\left|F_{j_\mu}\right|^2 + \lambda_{j_\mu} - 2 > \log|F_{j_\nu}(x)|^2 + \lambda_{j_\nu}$$

on $W \cap V_{j_\nu}$ for $1 \leq \mu \leq d$ and $d + 1 \leq \nu \leq m$. Thus

$$\alpha = \mathcal{M}_d\left[\left(\log|F_{j_1}|^2 + \lambda_{j_1}\right), \dots, \left(\log|F_{j_d}|^2 + \lambda_{j_d}\right)\right]$$

on $W \setminus Z$.

Now we may also choose W so that we have a proper local holomorphic model (W, Φ, W') and, for each $\nu = 1, \dots, d$, a holomorphic mapping $F'_{j_\nu} \colon W' \to \mathbb{C}^{N_{j_\nu} + l_{j_\nu}}$ such that $F'_{j_\nu} \circ \Phi = (F_{j_\nu}, 0, \dots, 0)$ on W and such that the coordinate functions for F'_{j_ν} generate the ideal sheaf $\mathcal{I}_{\Phi(A \cap W)}$ at each point. Furthermore, we may assume that there is a C^∞ strictly plurisubharmonic function ρ on W' and C^∞ functions $\lambda'_{j_1}, \dots, \lambda'_{j_d}$ on W' such that, for each $\nu = 1, \dots, d$, we have $\lambda'_{j_\nu} \circ \Phi = \lambda_{j_\nu}$ on W and $\lambda'_{j_\nu} + \rho$ is plurisubharmonic on W'. Setting $\alpha' \equiv -\infty$ on $\Phi(A \cap W)$ and

$$\alpha' = \mathcal{M}_d \left[\left(\log \left| F'_{j_1} \right|^2 + \lambda'_{j_1} \right), \dots, \left(\log \left| F'_{j_d} \right|^2 + \lambda'_{j_d} \right) \right]$$

on $W' \setminus \Phi(A \cap W)$, we see that $\alpha' \circ \Phi = \alpha$ on W, α' is C^∞ on $W' \setminus \Phi(A \cap W)$, and α' is continuous as a mapping into $[-\infty, \infty)$. Moreover, by part (i) of Lemma 4.3, $\alpha' + \rho$ is plurisubharmonic on $W' \setminus \Phi(A \cap W)$ and, therefore, on W'. It follows that α is of class $\mathcal{Q}^\infty_A$ on X. $\square$

Proof of Proposition 4.2. We may fix finite coverings $\{Q_i\}_{i=1}^k$, $\{U_i\}_{i=1}^k$, and $\{V_i\}_{i=1}^k$ of Y by nonempty connected open subsets of X and functions $\{\lambda_i\}_{i=1}^k$ such that, for each $i = 1, \dots, k$,

(4.2.1) Q_i is contractible;

(4.2.2) $V_i \Subset U_i \Subset Q_i \Subset \Omega$;

(4.2.3) We have $\operatorname{diam}_g U_i < \operatorname{dist}_g(U_i, X \setminus Q_i)$ (where the diameter diam_g is taken with respect to the distance function $\operatorname{dist}_g(\cdot, \cdot)$ in X;

(4.2.4) If $\overline{U_i} \cap \overline{U_j} \neq \emptyset$ for some j, then $U_i \Subset Q_j$;

(4.2.5) For every set A which is equal to a union of irreducible components of $Y \cap Q_i$ each of which meets U_i, we have a holomorphic map

$$F_{i,A} = \left(f_{i,A}^{(1)}, \dots, f_{i,A}^{(N_{i,A})} \right) : U_i \to \mathbb{C}^{N_{i,A}}$$

such that the coordinate functions generate the ideal sheaf $\mathcal{I}_A$ at each point in U_i and

$$|F_{i,A}|^2 = \sum_{j=1}^{N_{i,A}} |f_{i,A}^{(j)}|^2 \leq 1$$

(here, we allow for $A = \emptyset$ in which case we take $N_{i,A} = 1$ and $F_{i,A} = f_{i,A}^{(1)} \equiv 1$); and

(4.2.6) We have $\lambda_i \in C^\infty(V_i)$, $\lambda_i < -2$ on V_i, and $\lambda_i \to -\infty$ at ∂V_i.

We may also choose open sets Q_0, U_0, V_0, and W_0 in X with

$$X \setminus (V_1 \cup \cdots \cup V_k) \subset W_0 \subset \overline{W_0} \subset V_0 \subset \overline{V_0} \subset U_0 \subset \overline{U_0} \subset Q_0 \subset \overline{Q_0} \subset X \setminus Y$$

and a nonpositive function $\lambda_0 \in C^\infty(V_0)$ with $\lambda_0 \equiv 0$ on W_0 and $\lambda_0 \to -\infty$ at ∂V_0. We also set $N_{0,\emptyset} = 1$ and

$$F_{0,\emptyset} = f_{0,\emptyset}^{(1)} = f_{0,\emptyset}^{(N_{0,\emptyset})} \equiv 1.$$

For $i = 0, 1, 2, \dots, k$, let $\mathcal{A}_i$ be the (finite) collection of analytic subsets of Q_i which are either empty or the union of a set of irreducible components of $Y \cap Q_i$ each of which meets U_i. Given an open subset Θ of X, let $\mathcal{A}_\Theta$ be the collection of $(k+1)$-tuples $A = (A_0, \dots, A_k) \in \mathcal{A}_0 \times \cdots \times A_k$ such that, for all $i, j = 0, \dots, k$, we have

$$A_i \cap U_i \cap U_j \cap \Theta = A_j \cap U_j \cap U_i \cap \Theta.$$

In particular, the associated set $\check{A} \equiv [(A_0 \cap U_0) \cup \cdots \cup (A_k \cap U_k)] \cap \Theta$ is a (properly embedded) analytic subset of Θ with $\check{A} \cap U_i = A_i \cap U_i \cap \Theta$ for $i = 0, \dots, k$. Thus for each $A = (A_0, \dots, A_k) \in \mathcal{A}_\Theta$, as in Lemma 4.4, we may define a continuous function $\alpha_{\Theta,A} \colon \Theta \to [-\infty, 0]$ of class $\mathcal{Q}^\infty_{\check{A}}$ by setting $\alpha_{\Theta,A} \equiv -\infty$ on $\check{A}$ and, for each $x \in \Theta \setminus \check{A}$, setting

$$\alpha_{\Theta,A}(x) = \mathcal{M}_d\left[\left(\log\left|F_{i_1,A_{i_1}}(x)\right|^2 + \lambda_{i_1}(x)\right), \dots, \left(\log\left|F_{i_d,A_{i_d}}(x)\right|^2 + \lambda_{i_d}(x)\right)\right];$$

where $i_1, \dots, i_d$ are the distinct indices with $x \in V_i$ if and only if $i \in \{i_1, \dots, i_d\}$. We have $\alpha_{\Theta,A} \le 0$ because, in the above notation, all of the entries are nonpositive and at most one is greater than -2; while, if $t = (t_1, \dots, t_d)$ with $t_1 \le 0$ and $t_j < -2$ for $j = 2, \dots, d$, then

$$\mathcal{M}_d(t) \le \mathcal{M}_d(0, -2, \dots, -2) = \mathcal{M}_1(0) = 0.$$

Similarly, we have $\alpha_{\Theta,A} \equiv 0$ on $W_0 \cap \Theta \supset \Theta \setminus \Omega$. Since $X \setminus W_0 \subset \Omega \Subset X$, it follows that, given an open set V with $\overline{V} \subset \Theta$, there exists a positive constant $R_{\Theta,A,V}$ and nondecreasing continuous functions

$$\eta_{\Theta,A,V,-} \colon [0, \infty] \to [-\infty, 0] \qquad \text{and} \qquad \eta_{\Theta,A,V,+} \colon [0, \infty] \to [-\infty, 0]$$

such that $\eta_{\Theta,A,V,-} \le \eta_{\Theta,A,V,+}$, $\eta_{\Theta,A,V,+}(0) = -\infty$, $\eta_{\Theta,A,V,-} > -\infty$ on $(0, \infty]$,

$$\eta_{\Theta,A,V,-}(\operatorname{dist}_g(x, \check{A})) \le \alpha_{\Theta,A}(x) \le \eta_{\Theta,A,V,+}(\operatorname{dist}_g(x, \check{A})) \qquad \forall\, x \in V,$$

and, for each $\nu = 1, \dots, l$, the function $\alpha_{\Theta,A} + R_{\Theta,A,V} \cdot \rho_\nu$ is of class $\mathcal{SP}^\infty_{\check{A}}$ on $D_\nu \cap V$.

The collection $\mathcal{A}_{U_i} \subset \mathcal{A}_0 \times \cdots \times \mathcal{A}_k$ is finite for each i. Thus we may choose a constant $R_0 > 0$ and nondecreasing continuous functions

$$\eta_- \colon [0, \infty] \to [-\infty, 0] \qquad \text{and} \qquad \eta_+ \colon [0, \infty] \to [-\infty, 0]$$

such that $\eta_- \leq \eta_+$, $\eta_+(0) = -\infty$, $\eta_- > -\infty$ on $(0,\infty]$, and, for each $i = 0, 1, \dots, k$ and each $A \in \mathcal{A}_{U_i}$, we have $R_0 > R_{U_i,A,V_i}$, $\eta_- \leq \eta_{U_i,A,V_i,-} \leq \eta_{U_i,A,V_i,+} \leq \eta_+$, and $\eta_+ \equiv 0$ on the interval $[\operatorname{dist}_g(V_i, X \setminus U_i), \infty]$ (note that $\operatorname{dist}_g(V_0, X \setminus U_0) > 0$ since $X \setminus U_0$ is compact).

Given a connected covering space $\Upsilon\colon \widehat{X} \to X$ and an analytic subset Z of X which is equal to a union of irreducible components of $\widehat{Y} = \Upsilon^{-1}(Y)$, we will construct the associated class $\mathcal{Q}_Z^\infty$ function $\alpha\colon \widehat{X} \to [-\infty, 0]$. For this, we assume that the covering is infinite. The proof for a finite covering is similar.

For each $i = 0, \dots, k$, we let $\widehat{Q}_i = \Upsilon^{-1}(Q_i)$, $\widehat{U}_i = \Upsilon^{-1}(U_i)$, and $\widehat{V}_i = \Upsilon^{-1}(V_i)$. We set $\widehat{W_0} = \Upsilon^{-1}(W_0)$. For each $i = 1, \dots, k$, we let $\left\{Q_i^{(\nu)}\right\}_{\nu=1}^\infty$, $\left\{U_i^{(\nu)}\right\}_{\nu=1}^\infty$, and $\left\{V_i^{(\nu)}\right\}_{\nu=1}^\infty$ denote the distinct connected components of $\widehat{Q}_i$, $\widehat{U}_i$, and $\widehat{V}_i$, respectively, with $V_i^{(\nu)} \Subset U_i^{(\nu)} \Subset Q_i^{(\nu)}$ for each ν. For $1 \leq i \leq k$ and $\nu \geq 1$, we let $\lambda_i^{(\nu)} = \lambda_i \circ \Upsilon \restriction_{V_i^{(\nu)}} \in C^\infty\left(V_i^{(\nu)}\right)$, we let $Z_i^{(\nu)}$ be the union of all irreducible components of $Z \cap Q_i^{(\nu)}$ meeting $U_i^{(\nu)}$, we let $A_i^{(\nu)} = \Upsilon\left(Z_i^{(\nu)}\right) \in \mathcal{A}_i$, and we let $F_i^{(\nu)} = F_{i,A_i^{(\nu)}} \circ \Upsilon \restriction_{U_i^{(\nu)}} \colon U_i^{(\nu)} \to \mathbb{C}^{N_{i,A_i^{(\nu)}}}$. We also set $Q_0^{(1)} = \widehat{Q}_0$, $U_0^{(1)} = \widehat{U}_0$, $V_0^{(1)} = \widehat{V}_0$, $W_0^{(1)} = \widehat{W}_0$, $\lambda_0^{(1)} = \lambda_0 \circ \Upsilon \in C^\infty\left(V_0^{(1)}\right)$, $Z_0^{(1)} = \emptyset$, $A_0^{(1)} = \emptyset$, and $F_0^{(1)} = F_{0,\emptyset} \circ \Upsilon \equiv 1$ on $U_0^{(1)}$.

Lemma 4.4 applied to the above objects yields a function $\alpha\colon X \to [-\infty, 0]$ satisfying the conditions (i) and (ii). Specifically, $\alpha \equiv -\infty$ on Z while, for $x \in \widehat{X} \setminus Z$ and for distinct pairs of indices $(i_1, \nu_1), \dots, (i_d, \nu_d)$ with $x \in V_i^{(\nu)}$ if and only if $(i, \nu) \in \{(i_1, \nu_1), \dots, (i_d, \nu_d)\}$, we have,

$$\alpha(x) = \mathcal{M}_d\left[\left(\log\left|F_{i_1}^{(\nu_1)}(x)\right|^2 + \lambda_{i_1}^{(\nu_1)}(x)\right), \dots, \left(\log\left|F_{i_d}^{(\nu_d)}(x)\right|^2 + \lambda_{i_d}^{(\nu_d)}(x)\right)\right].$$

We will show that α satisfies the conditions (iii) (for $R > R_0$) and (iv) by showing that, locally, we have $\alpha = \alpha_{U_i,A} \circ \Upsilon$ with $A \in \mathcal{A}_{U_i}$.

Given $i \in \{1, \dots, k\}$ and $\nu \in \mathbb{N}$, we have *distinct* indices $i_1, \dots, i_m \in \{0, \dots, k\}$ and indices $\nu_1, \dots, \nu_m \in \mathbb{N}$ such that, for any pair of indices (j, μ), we have

$$U_i^{(\nu)} \cap U_j^{(\mu)} \neq \emptyset \iff (j, \mu) \in \{(i_1, \nu_1), \dots, (i_m, \nu_m)\}$$

(here we have used the condition (4.2.4)). In particular, $(i, \nu) \in \{(i_1, \nu_1), \dots, (i_m, \nu_m)\}$. For each $j = 0, \dots, k$, let $A_j \in \mathcal{A}_j$ be the analytic set given by

$$A_j = \begin{cases} \emptyset \text{ if } j \notin \{i_1, \dots, i_m\} \\ A_{i_s}^{(\nu_s)} \text{ if } j = i_s \end{cases}$$

Then the k-tuple $A \equiv (A_0, \dots, A_k)$ is an element of $\mathcal{A}_{U_i}$ with associated analytic set

$$\check{A} = \Upsilon\left(Z \cap U_i^{(\nu)}\right) = \Upsilon\left(Z_i^{(\nu)} \cap U_i^{(\nu)}\right) = A_i^{(\nu)} \cap U_i = A_i \cap U_i.$$

To see this, suppose $0 \leq j \leq k$. If $j = 0$ or $j \notin i_1, \ldots, i_m$, then $A_j \cap U_j \cap U_i = \emptyset = A_i \cap U_i \cap U_j$. If $j = i_s \neq 0$ for some s, then we have

$$A_j \cap U_j \cap U_i = A_{i_s}^{(\nu_s)} \cap U_{i_s} \cap U_i = \Upsilon\left(Z_{i_s}^{(\nu_s)} \cap U_{i_s}^{(\nu_s)}\right) \cap U_i = \Upsilon\left(Z \cap U_{i_s}^{(\nu_s)} \cap U_i^{(\nu)}\right);$$

where the last equality holds because $U_i^{(\nu)}$ is the unique component of $\widehat{U}_i$ meeting $U_{i_s}^{(\nu_s)}$. Exchanging j and i, we see that $A_j \cap U_j \cap U_i = A_i \cap U_i \cap U_j$ in this case as well and the claim follows.

Furthermore, we have $\alpha = \alpha_{U_i,A} \circ \Upsilon$ on $U_i^{(\nu)}$. For both functions equal $-\infty$ on $Z \cap U_i^{(\nu)}$. Given a point $x \in U_i^{(\nu)} \setminus Z$, we may rearrange the indices so that, for some $d \in 1, \ldots, m$, we have $x \in V_{i_1}^{(\nu_1)} \cap \cdots \cap V_{i_d}^{(\nu_d)}$ and $x \notin \widehat{V}_j$ for $j \notin \{i_1, \ldots, i_d\}$. Hence

$$\alpha(x) = \mathcal{M}_d\left[\left(\log\left|F_{i_1}^{(\nu_1)}(x)\right|^2 + \lambda_{i_1}^{(\nu_1)}(x)\right), \ldots, \left(\log\left|F_{i_d}^{(\nu_d)}(x)\right|^2 + \lambda_{i_d}^{(\nu_d)}(x)\right)\right].$$

We also have $y = \Upsilon(x) \in V_{i_1} \cap \cdots \cap V_{i_d}$ and $y \notin V_j$ for $j \notin \{i_1, \ldots, i_d\}$. Thus

$$\alpha_{U_i,A}(y) = \mathcal{M}_d\left[\left(\log\left|F_{i_1,A_{i_1}}(y)\right|^2 + \lambda_{i_1}(y)\right), \ldots, \left(\log\left|F_{i_d,A_{i_d}}(y)\right|^2 + \lambda_{i_d}(y)\right)\right];$$

and we get the claim.

It follows that the condition (iii) holds on the set $V_i^{(\nu)}$ for $R > R_0$. That is, for $R > R_0$, the function $\alpha + R \cdot \rho_\nu \circ \Upsilon$ is of class $\mathcal{SP}_Z^\infty$ on $V_i^{(\nu)} \cap \Upsilon^{-1}(D_\mu)$ for $\mu = 1, \ldots, l$. To verify that the condition (iv) also holds on $V_i^{(\nu)}$, we fix $x \in V_i^{(\nu)}$ and set $y = \Upsilon(x)$ and $r = \operatorname{dist}_{\hat{g}}(x, Z)$, where $\hat{g} = \Upsilon^* g$. The condition (4.2.3) ensures that $\operatorname{dist}_{\hat{g}}(a, b) = \operatorname{dist}_g(\Upsilon(a), \Upsilon(b))$ for all $a, b \in U_i^{(\nu)}$ and

$$\operatorname{diam}_{\hat{g}} U_i^{(\nu)} = \operatorname{diam}_g U_i < \operatorname{dist}_g(U_i, X \setminus Q_i) = \operatorname{dist}_{\hat{g}}\left(U_i^{(\nu)}, \widehat{X} \setminus Q_i^{(\nu)}\right).$$

In particular, we have

$$\begin{aligned} \eta_-(r) &\leq \eta_-\left(\operatorname{dist}_{\hat{g}}\left(x, Z \cap U_i^{(\nu)}\right)\right) = \eta_-(\operatorname{dist}_g(y, \check{A})) \\ &\leq \eta_{U_i,A,V_i,-}(\operatorname{dist}_g(y, \check{A})) \leq \alpha_{U_i,A}(y) = \alpha(x) \end{aligned}$$

For the other required inequality, we observe that, if

$$r \geq \operatorname{dist}_{\hat{g}}\left(V_i^{(\nu)}, \widehat{X} \setminus U_i^{(\nu)}\right) = \operatorname{dist}_g(V_i, X \setminus U_i),$$

then, by the choice of η_+, we have

$$\alpha(x) = \alpha_{U_i,A}(y) \leq 0 = \eta_+(r).$$

If $r < \operatorname{dist}_{\hat{g}}(V_i^{(\nu)}, \widehat{X} \setminus U_i^{(\nu)})$, then we must have

$$r = \operatorname{dist}_{\hat{g}}\left(x, Z \cap U_i^{(\nu)}\right) = \operatorname{dist}_g(y, \check{A})$$

and hence

$$\alpha(x) = \alpha_{U_i,A}(y) \leq \eta_{U_i,A,V_i,+}(r) \leq \eta_+(r).$$

Thus the condition (iv) holds on $V_i^{(\nu)}$.

It remains to show that the conditions (iii) and (iv) hold at points in $\widehat{X} \setminus (\widehat{V}_1 \cup \cdots \cup \widehat{V}_k) \subset \widehat{W}_0$. But $\alpha \equiv 0$ on $\widehat{W}_0$, so the condition (iii) holds for $R > R_0$ (or even $R > 0$) and we have $\eta_-(\mathrm{dist}_{\hat{g}}(\cdot, Z)) \leq \alpha$. Moreover, on $\widehat{W}_0$, we have

$$\mathrm{dist}_{\hat{g}}(\cdot, Z) \geq \mathrm{dist}_{\hat{g}}(\cdot, \widehat{Y}) \geq \mathrm{dist}_{\hat{g}}(\widehat{V}_0, \widehat{X} \setminus \widehat{U}_0) = \mathrm{dist}_g(V_0, X \setminus U_0).$$

and hence

$$\alpha = 0 = \eta_+(\mathrm{dist}_{\hat{g}}(\cdot, Z)).$$

Thus α satisfies the conditions (i)–(iv) (for $R > R_0$) on the entire covering space $\widehat{X}$. □

5. An $\mathcal{SP}^\infty(g,q)$ function near the singular set

For the proof of Theorem 0.1, Proposition 3.1 allows one to construct a function which, in a neighborhood of $\widehat{Y}$, is strongly q-convex away from $\widehat{Y}_{\mathrm{sing}}$. In order to modify the function to be strongly q-convex near the singular set, we will apply the following fact; the proof of which is the first goal of this section:

Proposition 5.1. *Let (X, g) be a connected reduced Hermitian complex space, let q be a positive integer, let Y be a compact analytic subset of X, let S be a compact analytic subset of dimension $< q$ in X, let Ω be a relatively compact neighborhood of S in X, let $\{D_\nu\}_{\nu=1}^m$ be relatively compact open subsets of X, and let ρ_ν be a C^∞ strictly plurisubharmonic function on a neighborhood of $\overline{D}_\nu$ for each $\nu = 1, \dots, m$. Then there is a relatively compact neighborhood Θ of S in Ω, a constant $R > 0$, and, for every $\epsilon > 0$, an associated $\delta > 0$ such that, for every connected covering space $\Upsilon \colon \widehat{X} \to X$ and for every analytic set Z equal to a union of irreducible components of $\widehat{Y} \equiv \Upsilon^{-1}(Y)$, there exists a C^∞ function α on $\widehat{X}$ with the following properties relative to the sets $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$ and $\widehat{\Theta} = \Upsilon^{-1}(\Theta)$ and the Hermitian metric $\hat{g} = \Upsilon^* g$:*

(i) *On $\widehat{X}$, $0 \leq \alpha \leq R$;*
(ii) *We have $\mathrm{supp}\,\alpha \subset \widehat{\Omega} \setminus N(Z; \delta)$ and $\alpha > 0$ on $\widehat{\Theta} \setminus N(Z; \epsilon)$;*
(iii) *On $\widehat{\Theta} \cup N(Z; \delta)$, α is of class $\mathcal{W}^\infty(\hat{g}, q)$;*
(iv) *On the set $\left\{ p \in \widehat{\Theta} \mid \alpha(p) > 0 \right\}$, α is of class $\mathcal{SP}^\infty(\hat{g}, q)$; and*
(v) *For each $\nu = 1, \dots, m$, $\alpha + R\rho_\nu \circ \Upsilon$ is C^∞ strictly plurisubharmonic on $\Upsilon^{-1}(D_\nu)$ and $R\alpha - \rho_\nu \circ \Upsilon$ is of class $\mathcal{SP}^\infty(\hat{g}, q)$ on a neighborhood of $\Upsilon^{-1}(D_\nu) \cap \widehat{\Theta} \setminus N(Z; \epsilon)$.*

Remark. Note that Θ and R do not depend on the choice of ϵ and δ.

Proof. We may assume without loss of generality that $\Omega \Subset D_1 \cup \cdots \cup D_m$. According to Theorem 2.4, there exists a nonnegative C^∞ function β on X such that $\operatorname{supp}\beta \subset \Omega$ and such that, on a neighborhood of the closure of some relatively compact neighborhood Θ of S in Ω, we have $\beta > 2$ and β is of class $\mathcal{SP}^\infty(g,q)$. We may also choose a C^∞ function $\chi\colon \mathbb{R} \to [0,\infty)$ such that $\chi \equiv 0$ on $(-\infty,0]$, $\chi' > 0$ and $\chi'' \geq 0$ on $(0,\infty)$, and $\chi'(1) \geq 1$. Choosing a constant $R_0 \gg 1$, we get $\beta \leq R_0$, $\chi(\beta) \leq R_0$, and $\chi'(\beta) \leq R_0$ on X. Choosing $R_1 \gg 0$, we get, for each $\nu = 1,\dots,m$, $R_1\beta - \rho_\nu$ is of class $\mathcal{SP}^\infty(g,q)$ on $D_\nu \cap \Theta$ and $R_0\beta + R_1\rho_\nu$ is C^∞ strictly plurisubharmonic on D_ν.

Applying Proposition 4.2, we may choose (independently of the choices made in the previous paragraph) a relatively compact neighborhood Λ of Y in X, a pair of nondecreasing continuous maps

$$\eta_-\colon [0,\infty] \to [-\infty,0] \qquad \text{and} \qquad \eta_+\colon [0,\infty] \to [-\infty,0]$$

with $\eta_- \leq \eta_+$, $\eta_+(0) = -\infty$, and $\eta_- > -\infty$ on $(0,\infty]$, and a constant $R_2 > 0$ such that, for every connected covering space $\Upsilon\colon \widehat{X} \to X$ and for every analytic set Z equal to a union of irreducible components of $\widehat{Y} \equiv \Upsilon^{-1}(Y)$, there exists a continuous function $\gamma\colon \widehat{X} \to [-\infty,0]$ with the following properties:

(5.1.1) The function γ is of class $\mathcal{Q}^\infty_Z$ on $\widehat{X}$;
(5.1.2) We have $\operatorname{supp}\gamma \subset \hat{\Lambda} \equiv \Upsilon^{-1}(\Lambda)$;
(5.1.3) For each $\nu = 1,\dots,m$, $\gamma + R_2 \cdot \rho_\nu \circ \Upsilon$ is of class $\mathcal{SP}^\infty_Z$ on $\Upsilon^{-1}(D_\nu)$;
(5.1.4) For $\hat{g} = \Upsilon^* g$, we have

$$\eta_-(\operatorname{dist}_{\hat{g}}(x,Z)) \leq \gamma(x) \leq \eta_+(\operatorname{dist}_{\hat{g}}(x,Z)) \qquad \forall\, x \in \widehat{X}.$$

We may now fix a constant $R > \max(R_0, 2R_1)$ and, given $\epsilon > 0$, we may choose a constant $\mu > 0$ so small that $2\mu R_1 R_2 < 2\mu R_0 R_1 R_2 < 1$, $R_1 + \mu R_0 R_2 < R$, and $\mu \cdot \eta_-(\epsilon) > -1$; and we may then choose a constant $\delta \in (0,\epsilon)$ so small that $\mu \cdot \eta_+(\delta) < -R_0$. We will show that, for $\Upsilon\colon \widehat{X} \to X$ a connected covering space, Z a union of irreducible components of $\widehat{Y} = \Upsilon^{-1}(Y)$, and $\gamma\colon \widehat{X} \to [-\infty,0]$ satisfying (5.1.1)–(5.1.4), the function α given by $\alpha \equiv 0$ on Z and $\alpha = \chi(\beta \circ \Upsilon + \mu\gamma)$ on $\widehat{X} \setminus Z$ has the required properties (i)–(v). For this, we set $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$, $\widehat{\Theta} = \Upsilon^{-1}(\Theta)$, $\hat{\beta} = \beta \circ \Upsilon$, and, for each $\nu = 1,\dots,m$, $\widehat{D}_\nu = \Upsilon^{-1}(D_\nu)$ and $\hat{\rho}_\nu = \rho_\nu \circ \Upsilon$ on $\widehat{D}_\nu$.

The choice of χ and R_0 guarantee that α is of class C^∞ and $0 \leq \alpha \leq \chi(\hat{\beta}) \leq R$ on $\widehat{X}$; as in (i). We also have $\hat{\beta} + \mu\gamma \leq 0$ on $\hat{\beta}^{-1}(0) \cup N(Z;\delta)$ and $\hat{\beta} + \mu\gamma \geq \hat{\beta} + \mu\eta_-(\epsilon) > 2 - 1 > 0$

on $\widehat{\Theta} \setminus N(Z;\epsilon)$, so the condition (ii) holds. For each $\nu = 1,\dots,m$, we have, on the set $\widehat{\Theta} \cap \widehat{D}_\nu \setminus Z$,

$$\hat{\beta} + \mu\gamma = R_1^{-1} \cdot (R_1\hat{\beta} - \hat{\rho}_\nu) + \mu \cdot (\gamma + R_2\hat{\rho}_\nu) + (R_1^{-1} - \mu R_2) \cdot \hat{\rho}_\nu.$$

Since $\chi', \chi'' \geq 0$ and $\alpha \equiv 0$ on $N(Z;\delta)$, Lemma 2.5 implies that α satisfies the condition (iii). Furthermore, since $\chi'(t) > 0$ precisely when $\chi(t) > 0$ (i.e. when $t > 0$), the condition (iv) holds.

It remains to verify the condition (v). Given an index ν with $1 \leq \nu \leq m$ and a point $p \in \widehat{D}_\nu$, we may choose a proper local holomorphic model (U, Φ, U') such that $p \in U \subset \widehat{D}_\nu$ and such that, for some choice of C^∞ strictly plurisubharmonic functions τ and ρ on U' and some function θ of class $\mathcal{SP}^\infty_{\Phi(Z\cap U)}$ on U', we have $R_0\hat{\beta} + R_1\hat{\rho}_\nu = \tau \circ \Phi$, $\hat{\rho}_\nu = \rho \circ \Phi$, and $\gamma + R_2 \cdot \hat{\rho}_\nu = \theta \circ \Phi$ on U. In particular, we get the extensions

$$\beta_0 \equiv R_0^{-1} \cdot (\tau - R_1\rho) \quad \text{and} \quad \gamma_0 \equiv \theta - R_2 \cdot \rho$$

of $\hat{\beta} \restriction_U$ and $\gamma \restriction_U$, respectively. If $p \in \widehat{\Theta} \cap \widehat{D}_\nu$, then we may also choose the local model so that $U \subset \widehat{\Theta}$ and so that, for some function $\omega \in C^\infty(U')$, we have $R_1\hat{\beta} - \hat{\rho}_\nu = \omega \circ \Phi$ on U and the $\hat{g}$-trace of the restriction of $\Phi^*\mathcal{L}(\omega)$ to any q-dimensional subspace of $\mathcal{T}^{(1)}_x U$ is positive for each $x \in U$. We then get a second extension $\beta_1 \equiv R_1^{-1} \cdot (\omega + \rho)$ of $\hat{\beta} \restriction_U$.

We also have a C^∞ extension φ of the function $(\alpha + R\hat{\rho}_\nu) \restriction_U$ given by

$$\varphi = \begin{cases} R\rho & \text{on } \Phi(Z \cap U) \\ \chi(\beta_0 + \mu\gamma_0) + R\rho & \text{on } U' \setminus \Phi(Z \cap U) \end{cases}$$

In particular, $\varphi = R\rho$ near $\Phi(Z\cap U)$ and hence φ is strictly plurisubharmonic near $\Phi(Z\cap U)$. For each point $x \in \Phi(U \setminus Z)$ and each nonzero tangent vector $v \in T^{1,0}_x U'$, we have

$$\mathcal{L}(\varphi)(v,v) \geq \chi'\left[\beta_0(x) + \mu\gamma_0(x)\right] \cdot \mathcal{L}(\beta_0 + \mu\gamma_0)(v,v) + R \cdot \mathcal{L}(\rho)(v,v).$$

Hence $\mathcal{L}(\varphi)(v,v) > 0$ if $\mathcal{L}(\beta_0 + \mu\gamma_0)(v,v) \geq 0$. If $\mathcal{L}(\beta_0 + \mu\gamma_0)(v,v) < 0$, then, since $0 \leq \chi'(\beta_0(x) + \mu\gamma_0(x)) \leq \chi'(\beta_0(x)) \leq R_0$, we get

$$\begin{aligned}
\mathcal{L}(\varphi)(v,v) &\geq R_0 \cdot \mathcal{L}(\beta_0 + \mu\gamma_0)(v,v) + R \cdot \mathcal{L}(\rho)(v,v) \\
&= R_0 \cdot \mathcal{L}\big(R_0^{-1} \cdot (\tau - R_1\rho) + \mu \cdot (\theta - R_2 \cdot \rho)\big)(v,v) + R \cdot \mathcal{L}(\rho)(v,v) \\
&= \mathcal{L}(\tau)(v,v) + \mu R_0 \cdot \mathcal{L}(\theta)(v,v) + (R - R_1 - \mu R_0 R_2) \cdot \mathcal{L}(\rho)(v,v) \\
&> 0.
\end{aligned}$$

It follows that φ is strictly plurisubharmonic on a neighborhood of $\Phi(U)$.

If $p \in \widehat{\Theta} \cap D_\nu$, ψ is the C^∞ extension of the function $(R\alpha - \hat{\rho}_\nu) \restriction_{U\setminus Z}$ to $U' \setminus \Phi(Z \cap U)$ given by $\psi \equiv R \cdot \chi(\beta_1 + \mu\gamma_0) - \rho$, $y \in U \setminus Z$, $x = \Phi(y)$, $e_1, \dots, e_q \in \mathcal{T}^{(1)}_y X$ are orthonormal,

and $v_j = \Phi_* e_j$ for $j = 1, \dots, q$, then

$$\begin{aligned}\sum_{j=1}^{q} \mathcal{L}(\psi)(v_j, v_j) \geq R \cdot \chi'\left(\hat{\beta}(y) + \mu\gamma(y)\right) \cdot \sum_{j=1}^{q} \mathcal{L}(\beta_1 + \mu\gamma_0)(v_j, v_j) - \sum_{j=1}^{q} \mathcal{L}(\rho)(v_j, v_j) \\ = RR_1^{-1} \cdot \chi'\left(\hat{\beta}(y) + \mu\gamma(y)\right) \cdot \sum_{j=1}^{q} \mathcal{L}(\omega)(v_j, v_j) \\ + \mu R \cdot \chi'\left(\hat{\beta}(y) + \mu\gamma(y)\right) \cdot \sum_{j=1}^{q} \mathcal{L}(\theta)(v_j, v_j) \\ + \left[R(R_1^{-1} - \mu R_2) \cdot \chi'\left(\hat{\beta}(y) + \mu\gamma(y)\right) - 1\right] \cdot \sum_{j=1}^{q} \mathcal{L}(\rho)(v_j, v_j)\end{aligned}$$

We have $R_1^{-1} - \mu R_2 = R_1^{-1}(1 - \mu R_1 R_2) > (2R_1)^{-1}$. If $y \in U \setminus N(Z; \epsilon)$, then $\hat{\beta}(y) + \mu\gamma(y) > 2 + \mu \cdot \eta_-(\epsilon) > 1$ and hence

$$R(R_1^{-1} - \mu R_2) \cdot \chi'(\hat{\beta}(y) + \mu\gamma(y)) - 1 > (R/(2R_1)) - 1 > 0.$$

Thus $\sum \mathcal{L}(\psi)(v_j, v_j) > 0$ for any choice of y in a small neighborhood of $U \setminus N(Z; \epsilon)$ and hence α and R satisfy the condition (v). $\square$

Combining Proposition 3.1 and Proposition 5.1, we get the following:

Proposition 5.2. *Let (X, g) be a connected reduced Hermitian complex space; let q be a positive integer; let Y be a compact analytic set of dimension $\leq q$; let C and S be analytic subsets of Y such that C is a union of irreducible components of Y, $\dim S < q$, S contains Y_{sing} as well as every irreducible component of Y of dimension $< q$, and $Y \setminus (C \cup S)$ is Stein; let $\{D_\nu\}_{\nu=1}^{m}$ be relatively compact open subsets of X; let ρ_ν be a C^∞ strictly plurisubharmonic function on a neighborhood of $\overline{D}_\nu$ for each $\nu = 1, \dots, m$; and let $\epsilon > 0$. Then, for every choice of constants δ_1, δ_2, δ_3, and δ_4 with $\epsilon \gg \delta_4 \gg \delta_3 \gg \delta_2 \gg \delta_1 > 0$ (i.e. one must choose δ_4 sufficiently small relative to ϵ, δ_3 sufficiently small relative to δ_4, and so on), there exists a neighborhood Ω of Y in X such that, for every connected covering space $\Upsilon \colon \widehat{X} \to X$, for every analytic set Z which is equal to a union of irreducible components of $\widehat{Y} \equiv \Upsilon^{-1}(Y)$ and which contains $\widehat{C} = \Upsilon^{-1}(C)$ as well as every q-dimensional compact irreducible component of $\widehat{Y}$ (i.e. Z contains every compact irreducible component of $\widehat{Y}$ not contained in $\widehat{S} = \Upsilon^{-1}(S)$), and for every positive continuous function θ on $\widehat{X}$, there exists a nonnegative C^∞ function α on $\widehat{X}$ with the following properties relative to the Hermitian metric $\hat{g} = \Upsilon^* g$, the sets $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$ and $\widehat{D}_\nu = \Upsilon^{-1}(D_\nu)$ for $\nu = 1, \dots, m$, and the functions $\hat{\rho}_\nu = \rho_\nu \circ \Upsilon \colon \widehat{D}_\nu \to \mathbb{R}$ for $\nu = 1, \dots, m$:*

(i) *On* $N(Z;\delta_1)\cap\widehat{\Omega}$, $\alpha\equiv 0$*;*

(ii) *On* $\left(\widehat{\Omega}\setminus N(Z;\delta_2)\right)\cup\left(\widehat{\Omega}\setminus\left[N(\widehat{S};\delta_2)\cup N(Z;\delta_1)\right]\right)$, $\alpha>0$*;*

(iii) *On* $N(\widehat{S};\delta_3)\cap\widehat{\Omega}$, α *is of class* $\mathcal{W}^\infty(\hat{g},q)$*;*

(iv) *On* $\left\{\,p\in N(\widehat{S};\delta_3)\cap\widehat{\Omega}\mid\alpha(p)>0\,\right\}$, α *is of class* $\mathcal{SP}^\infty(\hat{g},q)$*;*

(v) *For each* $\nu=1,\dots,m$, *the function* $\alpha-\hat{\rho}_\nu$ *is of class* $\mathcal{SP}^\infty(\hat{g},q)$ *on a neighborhood of* $N(\widehat{S};\delta_3)\cap\widehat{D}_\nu\cap\widehat{\Omega}\setminus N(Z;\delta_2)$*;*

(vi) *For each* $\nu=1,\dots,m$, *the function* $\alpha-\hat{\rho}_\nu$ *is* C^∞ *strongly* q*-convex on a neighborhood of* $\widehat{D}_\nu\cap\widehat{\Omega}\setminus\left[N(\widehat{S};\delta_2)\cup N(Z;\delta_1)\right]$*;*

(vii) *On* $\widehat{\Omega}\setminus\left[N(\widehat{S};\delta_4)\cup N(Z;\delta_1)\right]$, $\alpha>\theta$*; and*

(viii) *On* $\left\{\,p\in\widehat{\Omega}\mid\alpha(p)>0\,\right\}$, α *is* C^∞ *strongly* q*-convex.*

Proof. We may assume without loss of generality that $Y\subset\Theta_0\Subset D_1\cup\cdots\cup D_m$ for some open set Θ_0. We will also assume that $Y\setminus(C\cup S)\neq\emptyset$, since the proof for $Y\subset C\cup S$ is similar (but easier). Applying Proposition 5.1, we get a relatively compact neighborhood Θ_1 of S in Θ_0, a constant $R_0>1$, and, for every $\eta>0$, an associated $\delta>0$ such that, for every connected covering space $\Upsilon\colon\widehat{X}\to X$ and for every analytic set Z equal to a union of irreducible components of $\widehat{Y}\equiv\Upsilon^{-1}(Y)$, there exists a nonnegative C^∞ function β on $\widehat{X}$ with the following properties relative to the Hermitian metric $\hat{g}=\Upsilon^*g$, the sets $\widehat{\Theta}_j=\Upsilon^{-1}(\Theta_j)$ for $j=0,1$ and $\widehat{D}_\nu=\Upsilon^{-1}(D_\nu)$ for $\nu=1,\dots,m$, and the functions $\hat{\rho}_\nu=\rho_\nu\circ\Upsilon\colon\widehat{D}_\nu\to\mathbb{R}$ for $\nu=1,\dots,m$:

(5.2.1) We have $\operatorname{supp}\beta\subset\widehat{\Theta}_0\setminus N(Z;\delta)$ and $\beta>0$ on $\widehat{\Theta}_1\setminus N(Z;\eta)$;

(5.2.2) On $\widehat{\Theta}_1\cup N(Z;\delta)$, β is of class $\mathcal{W}^\infty(\hat{g},q)$;

(5.2.3) On the set $\left\{\,p\in\widehat{\Theta}_1\mid\beta(p)>0\,\right\}$, β is of class $\mathcal{SP}^\infty(\hat{g},q)$; and

(5.2.4) For each $\nu=1,\dots,m$, $\beta+R_0\hat{\rho}_\nu$ is C^∞ strictly plurisubharmonic on $\widehat{D}_\nu$ and $\beta-\hat{\rho}_\nu$ is of class $\mathcal{SP}^\infty(\hat{g},q)$ on a neighborhood of $\widehat{D}_\nu\cap\widehat{\Theta}_1\setminus N(Z;\eta)$ (in the notation of Proposition 5.1, $\beta=R\alpha$ and $R_0=R^2$).

We may now choose constants ξ, δ_2, δ_3, δ_4 and open sets $\Theta_2,\Omega_1,\dots,\Omega_4$ with the following properties:

(5.2.5) We have $0<2\xi<\delta_2<\delta_3<\delta_4<\min\left[\epsilon,\operatorname{dist}(Y,X\setminus\Theta_0),\operatorname{dist}(S,X\setminus\Theta_1)\right]$;

(5.2.6) We have $\Omega_4\Subset\cdots\Subset\Omega_1\Subset\Theta_0\setminus(C\cup S)$;

(5.2.7) For each connected component A of $Y\setminus(C\cup S)$ and each $j=1,\dots,4$, the set $\Omega_j\cap A$ is nonempty and connected, $\overline{\Omega_j\cap A}=\overline{\Omega}_j\cap A$, and

$$\pi_1(\Omega_j\cap A)\twoheadrightarrow\operatorname{im}\left[\pi_1(A)\to\pi_1(\bar{A})\right];$$

(5.2.8) $Y \setminus (C \cup S) \subset \Theta_2 \subset \Theta_0 \setminus (C \cup S)$, $\Theta_2 \setminus N(S; \delta_4) \subset \Omega_4$, $\Omega_3 \cap N(C \cup S; 2\delta_3) = \emptyset$, $\Theta_2 \setminus N(S; \delta_2) \subset \Omega_2$, and $\Omega_1 \cap N(C \cup S; 2\xi) = \emptyset$;

(5.2.9) For any connected covering space $\widehat{X}$ and any analytic set $Z \subset \widehat{X}$ as above, we may form a function β with the properties (5.2.1)–(5.2.4) for $\eta = \delta_2$ and $\delta = \xi$.

Note that we obtain the above sets and constants by choosing them in the order: δ_4, Ω_4, Ω_3, δ_3, δ_2, Ω_2, Ω_1, ξ, Θ_2. To get the surjection of fundamental groups in (5.2.7), we choose Ω_4 sufficiently large and apply standard facts (see, for example, Lemma 2 of [Fra]). To get ξ to satisfy (5.2.9), we need only choose η so that $0 < \eta \leq \delta_2$ and $\Omega_1 \cap N(C \cup S; \eta) = \emptyset$, and then choose ξ to be an associated δ as in (5.2.1)–(5.2.4) with $0 < 2\xi < \eta$. To get the condition (5.2.8), we first choose the constants ξ, δ_2, δ_3, δ_4 and the sets Ω_i for $i = 1, \dots, 4$ so that (5.2.8) holds with $Y \setminus (C \cup S)$ in place of Θ_2, and we then choose the neighborhood Θ_2 sufficiently small.

For each connected component Y' of $Y \setminus (C \cup S)$, we may form a connected neighborhood X' in Θ_2 with $\overline{X'} \subset \Theta_2 \cup S$ and $\pi_1(Y') \twoheadrightarrow \mathrm{im}\,[\pi_1(X') \to \pi_1(X)]$. We may also form a neighborhood Ξ' of Y' in X' and open sets $\Omega'_1, \dots, \Omega'_4$ with

$$X' \Supset \Omega'_1 \Supset \cdots \Supset \Omega'_4 \qquad \text{and} \qquad \Omega'_j \cap \Xi' = \Omega_j \cap \Xi' \text{ for } j = 1, \dots, 4.$$

By applying Proposition 3.1, we get a corresponding neighborhood as in part (b) (of Proposition 3.1) which we may take to be contained in Ξ' (note that X' is chosen so that each component $\widehat{X}'$ of the lifting in any covering space of X will contain exactly one component $\widehat{Y}'$ of the lifting of Y', so the proposition may be applied to the covering space $\widehat{X}' \to \widehat{Y}'$ whenever this restricted covering is infinite).

Taking the union of the (finitely many) resulting neighborhoods of all of the connected components of $Y \setminus (C \cup S)$ (which we may take to be disjoint), we get a neighborhood Θ_3 of $Y \setminus (C \cup S)$ in Θ_2 and an open subset B of Θ_3 such that $\overline{\Theta}_3 \subset \Theta_2 \cup S$, each connected component Θ' of Θ_3 meets (hence contains) exactly one connected component Y' of $Y \setminus (C \cup S)$ and satisfies

$$\pi_1(Y') \twoheadrightarrow \mathrm{im}\,[\pi_1(\Theta') \to \pi_1(X)]\,,$$

and, for every connected covering space $\Upsilon \colon \widehat{X} \to X$ and every positive continuous function θ on $\widehat{X}$, there is a nonnegative C^∞ function γ on the union Γ_0 of all of the connected components of $\widehat{\Theta}_3 = \Upsilon^{-1}(\Theta_3)$ which meet (hence contain) a connected component of $\Upsilon^{-1}(Y \setminus (C \cup S))$ which is not relatively compact in $\widehat{X}$ such that, if $\widehat{\Omega}_j = \Upsilon^{-1}(\Omega_j)$ for $j = 1, 2, 3, 4$, then

(5.2.10) On $\Gamma_0 \setminus \widehat{\Omega}_1$, $\gamma \equiv 0$;

(5.2.11) On $\widehat{\Omega}_2 \cap \Gamma_0$, $\gamma > 0$;

(5.2.12) There is a nonnegative function $\gamma_0 \in \mathcal{W}^\infty(g,q)(\Theta_3 \setminus \overline{\Omega}_4)$ such that $\gamma = \gamma_0 \circ \Upsilon$ on $\Gamma_0 \setminus \widehat{\Omega}_3$ and γ_0 is of class $\mathcal{SP}^\infty(g,q)$ on $\{\, x \in \Theta_3 \setminus \overline{\Omega}_4 \mid \gamma_0(x) > 0 \,\} \supset \Omega_2 \cap \Theta_3 \setminus \overline{\Omega}_4$;

(5.2.13) We have $B = (\Omega_3 \cap \Theta_3) \cup \{\, x \in \Theta_3 \setminus \Omega_3 \mid \gamma_0(x) > 0 \,\} \supset \Omega_2 \cap \Theta_3$ and, for $\widehat{B} = \Upsilon^{-1}(B)$, we have

$$\widehat{B} \cap \Gamma_0 = \{\, x \in \Gamma_0 \mid \gamma(x) > 0 \,\}$$

and, for any C^∞ function τ with compact support in B, the function $R \cdot \gamma + \tau \circ \Upsilon$ will be C^∞ strongly q-convex on $\widehat{B} \cap \Gamma_0$ for every sufficiently large positive constant R; and

(5.2.14) On $\widehat{\Omega}_4 \cap \Gamma_0$, $\gamma > \theta$.

Finally, we may choose a constant δ_1 such that $0 < \delta_1 < \xi$, $\operatorname{dist}(\Theta_3 \setminus N(S;\xi), C) > \delta_1$, and $\operatorname{dist}(Y \setminus (N(S;\xi) \cup C), X \setminus \Theta_3) > \delta_1$.

We will now show that any relatively compact neighborhood Ω of Y in the neighborhood $\Omega_0 \equiv N(Y;\delta_1) \cap [\Theta_3 \cup N(C \cup S;\delta_1)]$ and the constants δ_1, δ_2, δ_3, and δ_4 have the required properties. For this, suppose $\Upsilon \colon \widehat{X} \to X$ is a connected covering space, θ is a positive continuous function on $\widehat{X}$, Z is a union of irreducible components of $\widehat{Y} = \Upsilon^{-1}(Y)$ which contains $\widehat{C} = \Upsilon^{-1}(C)$ as well as every q-dimensional compact irreducible component of $\widehat{Y}$, $\hat{g} = \Upsilon^* g$, $\widehat{\Theta}_i = \Upsilon^{-1}(\Theta_i)$ for $i = 0,1,2,3$, $\widehat{\Omega}_i = \Upsilon^{-1}(\Omega_i)$ for $i = 0,1,2,3,4$, $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$, and $\hat{\rho}_\nu = \rho_\nu \circ \Upsilon$ on $\widehat{D}_\nu \equiv \Upsilon^{-1}(D_\nu)$ for $\nu = 1,\dots,m$. We will assume that the covering is infinite, since the proof for a finite covering is similar (but easier). We may also assume that θ is an exhaustion function (since we may replace θ by an exhaustion function which is greater than θ). In the above notation, we may form a nonnegative C^∞ function β on $\widehat{X}$ satisfying the conditions (5.2.1)–(5.2.4) with $\eta = \delta_2$ and $\delta = \xi$, and we may form a nonnegative C^∞ function γ on $\Gamma_0 \subset \widehat{\Theta}_3$ satisfying the conditions (5.2.10)–(5.2.14). Observe that, for B and $\widehat{B} = \Upsilon^{-1}(B)$ as in (5.2.13), we have $\Omega \cap \Theta_3 \setminus N(S;\delta_2) \Subset U \Subset \Omega_2 \cap \Theta_3 \subset B$ for some open set U. For, if $p \in \Omega \cap \Theta_3 \setminus N(S;\delta_2)$, then $\operatorname{dist}(p,x) < \delta_1$ for some point $x \in Y$ and we get $\operatorname{dist}(x,S) > \delta_2 - \delta_1 > \xi$. Since $\delta_1 < \operatorname{dist}(\Theta_3 \setminus N(S;\xi), C)$, we also have $x \notin C$. Thus

$$\Omega \cap \Theta_3 \setminus N(S;\delta_2) \subset N(Y \setminus [N(S;\xi) \cup C];\delta_1) \setminus N(S;\delta_2) \Subset \Theta_3 \setminus N(S;\delta_2) \subset \Omega_2 \cap \Theta_3.$$

Setting $\widehat{U} = \Upsilon^{-1}(U)$, we see that, by replacing γ with a large multiple, we may assume that $\gamma - (R_0 + 1) \cdot \hat{\rho}_\nu$ is C^∞ strongly q-convex on $\widehat{D}_\nu \cap \widehat{U} \cap \Gamma_0$ for $\nu = 1,\dots,m$.

Let $\Gamma \subset \Gamma_0$ be the union of all of the connected components of $\widehat{\Theta}_3$ which meet (hence contain) a connected component of $\widehat{Y} \setminus (Z \cup \widehat{S})$. Observe that we get a well-defined nonnegative C^∞ function ζ on $\widehat{\Omega}_0$ by setting $\zeta = \beta + \gamma$ on $\Gamma \cap \widehat{\Omega}_0$ and β elsewhere in $\widehat{\Omega}_0$.

For

$$\widehat{\Omega}_0 \cap \partial\Gamma \subset N(\widehat{Y};\delta_1) \cap \left[\widehat{\Theta}_3 \cup N(\widehat{C} \cup \widehat{S};\delta_1)\right] \cap \partial\widehat{\Theta}_3 = N(\widehat{C} \cup \widehat{S};\delta_1) \cap \partial\widehat{\Theta}_3.$$

Since $\gamma \equiv 0$ on $\Gamma_0 \setminus \widehat{\Omega}_1 \supset N(\widehat{C} \cup \widehat{S};\delta_1) \cap \Gamma_0$, we see that ζ is C^∞. Cutting off, we get a nonnegative C^∞ function α on $\widehat{X}$ with $\alpha = \zeta$ on $\widehat{\Omega}$.

In order to verify the properties (i)–(viii), we first observe that we have the following:

$$\widehat{\Omega} \subset N(\widehat{Y};\delta_1) \subset N(\widehat{S};2\xi) \cup \Gamma \cup N(Z;\delta_1) \qquad \text{and} \qquad \Gamma \cap N(Z;\delta_1) \subset N(\widehat{S};2\xi).$$

For, given $x \in N(\widehat{Y};\delta_1) \setminus N(\widehat{S};2\xi)$, we may fix a point $y \in \widehat{Y}$ with $\operatorname{dist}(x,y) < \delta_1$. Hence $\operatorname{dist}\left(y,\widehat{S}\right) > 2\xi - \delta_1 > \xi$. If $y \in \widehat{C} \subset Z$, then, since $\operatorname{dist}\left[\widehat{\Theta}_3 \setminus N(\widehat{S};\xi),\widehat{C}\right] > \delta_1$, we get $x \in N(Z;\delta_1) \setminus \widehat{\Theta}_3 \subset N(Z;\delta_1) \setminus \Gamma$. If $y \notin \widehat{C}$, then, since $\operatorname{dist}\left[\widehat{Y} \setminus \left(N(\widehat{S};\xi) \cup \widehat{C}\right), \widehat{X} \setminus \widehat{\Theta}_3\right] > \delta_1$, a piecewise C^∞ path from y to x of length $< \delta_1$ must lie entirely in the connected component V of $\widehat{\Theta}_3$ containing y. Thus either $y \in Z$, in which case $x \in N(Z;\delta_1)$ and $x \in V \subset \widehat{\Theta}_3 \setminus \Gamma$; or $y \notin Z$, in which case $x \in V \subset \Gamma$.

Verification of (i). The condition (i) holds because $\gamma \equiv 0$ on $\Gamma \setminus \widehat{\Omega}_1 \supset \Gamma \cap N(\widehat{C} \cup \widehat{S};2\xi) \supset \Gamma \cap N(Z;\delta_1)$ and $\beta \equiv 0$ on $N(Z;\xi) \supset N(Z;\delta_1)$.

Verification of (ii). We have $\alpha \geq \beta > 0$ on $\widehat{\Theta}_1 \cap \widehat{\Omega} \setminus N(Z;\delta_2)$ and $\alpha \geq \gamma > 0$ on $\widehat{\Omega}_2 \cap \Gamma \cap \widehat{\Omega} \supset \Gamma \cap \widehat{\Omega} \setminus N(\widehat{S};\delta_2) \supset \Gamma \cap \widehat{\Omega} \setminus N(\widehat{S};\delta_4) \supset \Gamma \cap \widehat{\Omega} \setminus \widehat{\Theta}_1$. On the other hand, we have

$$\widehat{\Omega} \setminus \left(\Gamma \cup \widehat{\Theta}_1\right) \subset \widehat{\Omega} \setminus \left(\Gamma \cup N(\widehat{S};\delta_2)\right) \subset \widehat{\Omega} \setminus \left(\Gamma \cup N(\widehat{S};2\xi)\right) \subset N(Z;\delta_1).$$

That (ii) holds now follows.

Verification of (iii). From (5.2.2), we see that β is of class $\mathcal{W}^\infty(\hat{g},q)$ on $\widehat{\Theta}_1 \supset N(\widehat{S};\delta_3)$; and from (5.2.8) and (5.2.12), we see that γ is of class $\mathcal{W}^\infty(\hat{g},q)$ on $\Gamma \setminus \overline{\widehat{\Omega}_3} \supset \Gamma \cap N(\widehat{S};2\delta_3)$. It follows that the condition (iii) holds.

Verification of (iv). Suppose $p \in N(\widehat{S};\delta_3) \cap \widehat{\Omega}$ with $\alpha(p) = \zeta(p) > 0$. If $\beta(p) > 0$, then β is of class $\mathcal{SP}^\infty(\hat{g},q)$ near p (by (5.2.3)) and hence, since γ is of class $\mathcal{W}^\infty(\hat{g},q)$ on $\Gamma \cap N(\widehat{S};\delta_3)$, we see that α is of class $\mathcal{SP}^\infty(\hat{g},q)$ near p. If $\beta(p) = 0$, then we have $\gamma(p) > 0$ and

$$p \in N(\widehat{S};\delta_3) \cap \widehat{\Omega} \cap \Gamma \subset \widehat{\Theta}_1 \cap \Gamma_0 \setminus \overline{\widehat{\Omega}_3}.$$

Thus, near p, $\gamma = \gamma_0 \circ \Upsilon$ is of class $\mathcal{SP}^\infty(\hat{g},q)$ and β is of class $\mathcal{W}^\infty(\hat{g},q)$, and hence α is again of class $\mathcal{SP}^\infty(\hat{g},q)$ near p.

Verification of (v). Given $\nu \in \{1,\ldots,m\}$, the function $\beta - \hat{\rho}_\nu$ is of class $\mathcal{SP}^\infty(\hat{g},q)$ on a neighborhood of $\widehat{D}_\nu \cap \widehat{\Theta}_1 \setminus N(Z;\delta_2) \supset N(\widehat{S};\delta_3) \cap \widehat{D}_\nu \cap \widehat{\Omega} \setminus N(Z;\delta_2)$. Since γ is of class $\mathcal{W}^\infty(\hat{g},q)$ on $\Gamma \cap N(\widehat{S};\delta_3)$, the condition (v) holds.

Verification of (vi). Given $\nu \in \{1, \dots, m\}$, we have

$$\widehat{\Omega} \cap \widehat{D}_\nu \setminus \Big[N(\widehat{S}; \delta_2) \cup N(Z; \delta_1) \Big] \subset \Gamma \cap \widehat{U} \cap \widehat{\Omega} \cap \widehat{D}_\nu.$$

Moreover, on $\Gamma \cap \widehat{U} \cap \widehat{\Omega} \cap \widehat{D}_\nu$, we have $\alpha - \hat{\rho}_\nu = (\beta + R_0 \hat{\rho}_\nu) + (\gamma - (R_0 + 1) \cdot \hat{\rho}_\nu)$; the sum of a C^∞ strictly plurisubharmonic function and a C^∞ strongly q-convex function. Thus we get the property (vi).

Verification of (vii). We have

$$\widehat{\Omega} \setminus \Big[N(\widehat{S}; \delta_4) \cup N(Z; \delta_1) \Big] \subset \Gamma \cap \widehat{\Omega} \setminus N(\widehat{S}; \delta_4) \subset \Gamma \cap \widehat{\Omega} \cap \widehat{\Omega}_4.$$

Therefore, on this set, we have $\alpha \geq \gamma > \theta$.

Verification of (viii). Recall that we extended the list of functions $\{\rho_\nu\}$ so that $\widehat{\Omega} \Subset \Theta_0 \Subset D_1 \cup \cdots \cup D_m$. Hence the conditions (i), (iv), and (vi) give (viii). □

6. A uniformly quasi-plurisubharmonic exhaustion function

To obtain the desired exhaustion function, we will add to the function produced in Proposition 5.2 an exhaustion function which is uniformly quasi-plurisubharmonic and which is locally constant near the compact irreducible components of the lifting of the compact analytic set. Such a function is produced in the following proposition which is the main goal of this section:

Proposition 6.1. *Let (X, g) be a connected reduced Hermitian complex space, let Y be a compact analytic subset of X, and, for each $\nu = 1, \dots, m$, let D_ν be a relatively compact open subset of X and let ρ_ν be a C^∞ strictly plurisubharmonic function on a neighborhood of $\overline{D}_\nu$. Then there exist constants $R > 0$ and $\delta > 0$ such that, for every $\eta > 0$, for every set $E \subset \mathbb{R}$ with $|s - t| \geq 4\eta$ for all $s, t \in E$ with $s \neq t$, for every set $F \subset \mathbb{R}$ with $[0, \infty) \subset N_{\mathbb{R}}(F; \eta)$, for every connected covering space $\Upsilon \colon \widehat{X} \to X$, and for every analytic set Z which is equal to a union of irreducible components of $\widehat{Y} = \Upsilon^{-1}(Y)$ and which has only compact connected components, there exists a C^∞ positive exhaustion function τ on $\widehat{X}$ with the following properties:*

(i) *The function $\tau + R \cdot (1 + \eta) \cdot \rho_\nu \circ \Upsilon$ is C^∞ strictly plurisubharmonic on $\Upsilon^{-1}(D_\nu)$ for $\nu = 1, \dots, m$;*
(ii) *For $\hat{g} = \Upsilon^* g$, τ is locally constant on $N_{\hat{g}}(Z; \delta)$; and*
(iii) *We have $\tau(Z) \subset F \setminus N_{\mathbb{R}}(E; \eta)$.*

Lemma 6.2. *Let $\eta > 0$, let $E \subset \mathbb{R}$ with $|s-t| \geq 4\eta$ for all $s, t \in E$ with $s \neq t$, and let $F \subset \mathbb{R}$ with $[0, \infty) \subset N_{\mathbb{R}}(F; \eta)$. Then, for each $a \in [0, \infty)$, there exists a number $b \in [0, 6\eta)$ such that $a + b \in F \setminus N_{\mathbb{R}}(E; \eta)$.*

Proof. For some $c \in F$, we have $|a + \eta - c| < \eta$ and hence $a < c < a + 2\eta$. If $c \notin N_{\mathbb{R}}(E; \eta)$, then we may set $b = c - a$. If $|c - t| < \eta$ for some $t \in E$, then $t + 2\eta > 0$ and hence, for some $d \in F$, we have $|t + 2\eta - d| < \eta$. It follows that $d \notin N_{\mathbb{R}}(E; \eta)$, since $\eta < |t - d|$ and, for each $s \in E \setminus \{t\}$, $|s - d| \geq |s - t| - |t - d| > 4\eta - 3\eta = \eta$. Moreover,

$$a \leq c < t + \eta < d < t + 3\eta < c + 4\eta < a + 6\eta.$$

Thus $b = d - a$ has the required properties. □

Lemma 6.3. *Let $\{s_\nu\}$ and $\{t_\nu\}$ be sequences of nonnegative numbers with $t_\nu \to \infty$ as $\nu \to \infty$. Then there exists a Lipschitz continuous function χ on $\mathbb{R}$ such that*

(i) *We have $\chi > 2$ and $0 \leq \chi' \leq 1/2$;*
(ii) *We have $\chi(t) \to \infty$ as $t \to \infty$;*
(iii) *For all $s, t \in \mathbb{R}$ with $|s - t| < 1$, we have $\chi(s) < 2 \cdot \chi(t)$; and*
(iv) *For all $\nu \in \mathbb{N}$ and all $t \in \mathbb{R}$ with $t_\nu - 1 < t < s_\nu + t_\nu + 1$, we have*

$$4^{-1} \cdot \chi(t) < \chi(t_\nu) < 4 \cdot \chi(t).$$

Proof. Reordering if necessary, we may assume without loss of generality that $\{t_\nu\}$ is nondecreasing. We may choose a subsequence $\{t_{\nu_j}\}_{j=1}^{\infty}$ such that $t_{\nu_1} = t_1$ and such that, for each $j > 1$, $t_{\nu_j} > s_\nu + t_\nu$ for all $\nu \leq \nu_{j-1}$ and $t_{\nu_j} > t_{\nu_{j-1}} + 2$. We now let χ be the piecewise linear continuous function for which $\chi \equiv 3$ on $(-\infty, t_{\nu_1}]$ and, for $j > 1$, $\chi(t_{\nu_j}) = j + 2$ and $\chi'' \equiv 0$ on $(t_{\nu_{j-1}}, t_{\nu_j})$.

For each ν, there is a unique index j with $t_\nu \in [t_{\nu_{j-1}}, t_{\nu_j})$ and hence $j+1 \leq \chi(t_\nu) < j+2$. If $s_\nu + t_\nu \leq t_{\nu_j}$, then

$$\chi(s_\nu + t_\nu) \leq \chi(t_{\nu_j}) = j + 2 \leq \frac{j+2}{j+1}\chi(t_\nu).$$

Suppose $s_\nu + t_\nu > t_{\nu_j}$. Since $t_\nu < t_{\nu_j}$, we have $\nu < \nu_j$ and, therefore, $t_{\nu_{j+1}} > s_\nu + t_\nu$ by the choice of $t_{\nu_{j+1}}$. Thus

$$\chi(s_\nu + t_\nu) \leq \chi(t_{\nu_{j+1}}) = j + 3 \leq \frac{j+3}{j+1}\chi(t_\nu).$$

Thus $\chi(s_\nu + t_\nu) \leq 2\chi(t_\nu)$ for each ν.

Since χ is piecewise linear, for each $j > 1$ and each $t \in (t_{\nu_{j-1}}, t_{\nu_j})$, we have

$$\chi'(t) = \frac{1}{t_{\nu_j} - t_{\nu_{j-1}}} < \frac{1}{2}.$$

So (i) holds.

If $|s-t|<1$, then, since $\chi>2$ and $0\leq\chi'\leq 1/2$, we have

$$\chi(s)\leq\chi(t+1)\leq\chi(t)+\frac{1}{2}<2\cdot\chi(t).$$

Hence (iii) holds.

Finally, if $\nu\in\mathbb{N}$ and $t\in(t_\nu-1,s_\nu+t_\nu+1)$, then $|s-t|<1$ for some $s\in[t_\nu,s_\nu+t_\nu]$. Therefore, by the above,

$$\frac{1}{2}\cdot\chi(t)<\chi(s)\leq\chi(s_\nu+t_\nu)\leq 2\cdot\chi(t_\nu)$$

and

$$\chi(t_\nu)\leq\chi(s)<2\cdot\chi(t).$$

The property (iv) now follows. □

Proof of Proposition 6.1. We may fix a nonempty relatively compact neighborhood Ω of Y containing $\overline{D}_\nu$ for $\nu=1,\dots,m$. According to Proposition 1.1, there exists a Hermitian metric $g'\geq g$ on X such that $g'=g$ on Ω and $x\mapsto\operatorname{dist}_{g'}(x,p)$ is an exhaustion function. If we find a $\delta>0$ which works for this metric g' and which satisfies $\delta<\operatorname{dist}_g(Y,X\setminus\Omega)\leq\operatorname{dist}_{g'}(Y,X\setminus\Omega)$, then the δ-neighborhoods in coverings of liftings of subsets of Y will be the same for both metrics and hence this δ will also work for the metric g. Thus we may assume without loss of generality that $x\mapsto\operatorname{dist}_g(x,p)$ is an exhaustion function for each point $p\in X$.

We may first choose $\delta_0>0$ such that $\delta_0<1$, $\delta_0<\operatorname{dist}_g(Y,X\setminus\Omega)$, and, for each point $p\in\overline{\Omega}$, $B_g(p,\delta_0)$ is contained in some contractible open set. According to Lemma 1.3, we may also choose δ_0 so that any two disjoint irreducible components of the lifting of Y to a connected covering space are of distance $>\delta_0$ with respect to the lifting of g. We may now choose a number δ with $0<\delta<\delta_0/4$, a covering $V_1,\dots,V_k$ of $\overline{\Omega}$ by finitely many relatively compact connected open subsets of X each of which is of diameter $<\delta$ and meets Ω, and nonnegative C^∞ functions $\{\alpha_i\}_{i=1}^k$ such that $\operatorname{supp}\alpha_i\subset V_i$ for each i and $\sum\alpha_i\equiv 1$ on a neighborhood Θ of $\overline{\Omega}$ in X.

Suppose now that $\eta>0$, $E\subset\mathbb{R}$ with $|s-t|\geq 4\eta$ for all $s,t\in E$ with $s\neq t$, $F\subset\mathbb{R}$ with $[0,\infty)\subset N_{\mathbb{R}}(F;\eta)$, $\Upsilon\colon\widehat{X}\to X$ is a connected covering space, Z is an analytic subset of X which is equal to a union of irreducible components of $\widehat{Y}=\Upsilon^{-1}(Y)$ and which has only compact connected components, and $\hat{g}=\Upsilon^*g$. Fixing a point $O\in X$, Lemma 1.2 implies that the function $r\colon x\mapsto\operatorname{dist}_{\hat{g}}(x,O)$ is an exhaustion function. In order to produce the function τ, we may assume that the covering is infinite since, for a

finite covering, an exhaustion function which is equal to a suitable constant on the lifting of Ω will work. Let $\{Z_\sigma\}_{\sigma\in S}$ be the distinct connected components of Z. Observe that $\overline{N_{\hat{g}}(\widehat{Y};\delta_0)} \subset \widehat{\Omega} \equiv \Upsilon^{-1}(\Omega)$.

By the choice of δ_0 and δ, each of the compact sets $\overline{V}_i$ is contained in a contractible open set and is therefore evenly covered by Υ. For each i, we let $\left\{V_i^{(\nu)}\right\}_{\nu=1}^{\infty}$ be the connected components of $\widehat{V}_i = \Upsilon^{-1}(V_i)$. The collection $\left\{V_i^{(\nu)}\right\}$ is locally finite in $\widehat{X}$ and (again, by the choice of δ_0 and δ) each set $V_i^{(\nu)}$ is of diameter $< \delta$ (with respect to the metric $\operatorname{dist}_{\hat{g}}(\cdot,\cdot)$). For each $i = 1,\dots,k$ and each $\nu = 1,2,3,\dots$, we define the nonnegative C^∞ function $\alpha_i^{(\nu)}$ with compact support in $V_i^{(\nu)}$ by

$$\alpha_i^{(\nu)}(x) = \begin{cases} \alpha_i(\Upsilon(x)) & \text{if } x \in V_i^{(\nu)} \\ 0 & \text{if } x \in \widehat{X} \setminus V_i^{(\nu)} \end{cases}$$

We have $\sum_{i,\nu} \alpha_i^{(\nu)} \equiv 1$ on the set $\widehat{\Theta} = \Upsilon^{-1}(\Theta)$. By Lemma 6.3, there exists a Lipschitz continuous function $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $\chi > 2$, $0 \le \chi' \le 1/2$, $\chi(t) \to \infty$ as $t \to \infty$, $\chi(s) < 2\cdot\chi(t)$ whenever $s,t \in \mathbb{R}$ with $|s-t| < 1$, and

$$\frac{1}{4}\cdot\chi(t) < \chi\big[\operatorname{dist}_{\hat{g}}(O,Z_\sigma)\big] < 4\cdot\chi(t)$$

whenever $\sigma \in S$ and $\operatorname{dist}_{\hat{g}}(O,Z_\sigma) - 1 < t < \operatorname{diam}_{\hat{g}}(Z_\sigma) + \operatorname{dist}_{\hat{g}}(O,Z_\sigma) + 1$. Finally, for each $\sigma \in S$, let M_σ be the set of pairs of indices (i,ν) such that $1 \le i \le k$, $\nu \in \mathbb{N}$, and $V_i^{(\nu)} \cap N_{\hat{g}}(Z_\sigma;\delta) \neq \emptyset$. We also let M be the set of pairs of indices (i,ν) such that $1 \le i \le k$, $\nu \in \mathbb{N}$, and $V_i^{(\nu)} \cap N_{\hat{g}}(Z;\delta) = \emptyset$. Observe that the sets $\{M_\sigma\}$ are mutually disjoint and disjoint from M. For, if $V_i^{(\nu)}$ meets $N_{\hat{g}}(Z_\sigma;\delta)$ and $N_{\hat{g}}(Z_{\sigma'};\delta)$, then, since $\operatorname{diam}_{\hat{g}}\left(V_i^{(\nu)}\right) < \delta < \delta_0/4$, we have $\operatorname{dist}_{\hat{g}}(Z_\sigma, Z_{\sigma'}) < \delta_0$. Hence $Z_\sigma \cap Z_{\sigma'} \neq \emptyset$ and, therefore, $\sigma = \sigma'$.

We may now define a C^∞ function α on $\widehat{X}$ by

$$\alpha = \sum_{\sigma\in S}\sum_{(i,\nu)\in M_\sigma} \alpha_i^{(\nu)}\cdot\chi\big[\operatorname{dist}_{\hat{g}}(O,Z_\sigma)\big] + \sum_{(i,\nu)\in M} \alpha_i^{(\nu)}\cdot\chi\left[\operatorname{dist}_{\hat{g}}\left(O,V_i^{(\nu)}\right)\right].$$

We note that the previous remarks imply that the above sum is locally finite. Moreover, each pair (i,ν) appears exactly once and we have $\alpha > 2$ on $\widehat{\Theta} = \Upsilon^{-1}(\Theta)$. We will show that the function $\beta = \log\alpha$ on $\widehat{\Theta}$ has the following properties:

(6.1.1) On the δ-neighborhood $\overline{N_{\hat{g}}(Z;\delta)} \subset \widehat{\Omega} = \Upsilon^{-1}(\Omega)$ of Z, β is locally constant;

(6.1.2) The function β exhausts $\widehat{\Omega}$; and

(6.1.3) There is a constant $R_0 > 0$ which is independent of the choice of the covering space $\widehat{X}$ and of the analytic subset Z and for which the function $\beta + R_0 \cdot \rho_\nu \circ \Upsilon$ is C^∞ strictly plurisubharmonic on $\Upsilon^{-1}(D_\nu)$ for $\nu = 1, \dots, m$.

For the property (6.1.1), we need only observe that, for each $\sigma \in S$, we have $\alpha_i^{(\nu)} \equiv 0$ on $N_{\hat{g}}(Z_\sigma; \delta)$ for each $(i, \nu) \notin M_\sigma$. Thus, on $N(Z_\sigma; \delta)$,

$$\alpha = \sum_{(i,\nu) \in M_\sigma} \alpha_i^{(\nu)} \cdot \chi\Big[\operatorname{dist}_{\hat{g}}(O, Z_\sigma)\Big] \equiv \chi\Big[\operatorname{dist}_{\hat{g}}(O, Z_\sigma)\Big].$$

For (6.1.2), we first observe that, for each index $\sigma \in S$, each pair $(i, \nu) \in M_\sigma$, and each point $x \in V_i^{(\nu)}$, we have

$$\operatorname{dist}_{\hat{g}}(O, Z_\sigma) - 2\delta < r(x) = \operatorname{dist}_{\hat{g}}(O, x) < \operatorname{diam}_{\hat{g}}(Z_\sigma) + \operatorname{dist}_{\hat{g}}(O, Z_\sigma) + 2\delta.$$

For each pair $(i, \nu) \in M$ and each point $x \in V_i^{(\nu)}$, we have

$$\operatorname{dist}_{\hat{g}}(O, V_i^{(\nu)}) < r(x) < \operatorname{dist}_{\hat{g}}(O, V_i^{(\nu)}) + \delta.$$

Therefore, by the choice of χ, we have $4^{-1} \cdot \chi(r) \leq \alpha \leq 4 \cdot \chi(r)$ on $\widehat{\Theta}$. In particular, α and β exhaust $\overline{\widehat{\Omega}}$, so the property (6.1.2) holds.

Given a point $p \in \Theta$, we may choose a proper local holomorphic model (U, Φ, U'), a Hermitian metric g' on U', C^∞ functions $\{\alpha_i'\}_{i=1}^k$ on U', and a constant $A > 0$ such that $p \in U \subset \Theta$, $\Phi^* g' = g$ on U, and, for each $i = 1, \dots, k$, $\Phi^* \alpha_i' = \alpha_i$ on U, and $|d\alpha_i'|_{g'} \leq A$ and $-A \cdot g' \leq \mathcal{L}(\alpha_i') \leq A \cdot g'$ on U'. We may also assume that there is a constant $B > 0$ such that, whenever $U \cap D_\nu \neq \emptyset$, there is a C^∞ strictly plurisubharmonic function ρ_ν' on U' with $\rho_\nu = \rho_\nu' \circ \Phi$ on U and $B \cdot \mathcal{L}(\rho_\nu') \geq (16kA + 256k^2A^2 + 1) \cdot g'$ on U'.

For each point $a \in \widehat{U} \equiv \Upsilon^{-1}(U)$, there is a finite set N of pairs of indices (i, ν) such that $a \in V_i^{(\nu)}$ if and only if $(i, \nu) \in N$. In particular, for distinct elements $(i, \nu), (j, \mu) \in N$, we have $i \neq j$; so $\#N \leq k$. Thus, on some relatively compact neighborhood W of a in $\bigcap_{(i,\nu) \in N} V_i^{(\nu)} \cap \widehat{U}$, we have

$$\alpha = \sum_{\sigma \in S} \sum_{(i,\nu) \in N \cap M_\sigma} \alpha_i^{(\nu)} \cdot \chi\big[\operatorname{dist}_{\hat{g}}(O, Z_\sigma)\big] + \sum_{(i,\nu) \in N \cap M} \alpha_i^{(\nu)} \cdot \chi\left[\operatorname{dist}_{\hat{g}}\left(O, V_i^{(\nu)}\right)\right].$$

Setting $\Psi = \Phi \circ \Upsilon \restriction_W$ and

$$\alpha' = \sum_{\sigma \in S} \sum_{(i,\nu) \in N \cap M_\sigma} \alpha_i' \cdot \chi\big[\operatorname{dist}_{\hat{g}}(O, Z_\sigma)\big] + \sum_{(i,\nu) \in N \cap M} \alpha_i' \cdot \chi\left[\operatorname{dist}_{\hat{g}}\left(O, V_i^{(\nu)}\right)\right] \in C^\infty(U'),$$

we get $\Psi^*\alpha' = \alpha$ and $\Psi^* g' = \hat{g}$ on W and, for each point $z = \Psi(x)$ with $x \in W$, we have

$$\begin{aligned}
|(d\alpha')_z|_{g'} &= \left| \sum_{\sigma\in S} \sum_{(i,\nu)\in N\cap M_\sigma} (d\alpha_i')_z \cdot \chi\big[\mathrm{dist}_{\,\hat{g}}(O, Z_\sigma)\big] + \sum_{(i,\nu)\in N\cap M} (d\alpha_i')_z \cdot \chi\left[\mathrm{dist}_{\,\hat{g}}\left(O, V_i^{(\nu)}\right)\right] \right|_{g'} \\
&\leq \sum_{\sigma\in S} \sum_{(i,\nu)\in N\cap M_\sigma} |(d\alpha_i')_z|_{g'} \cdot \chi\big[\mathrm{dist}_{\,\hat{g}}(O, Z_\sigma)\big] \\
&\qquad + \sum_{(i,\nu)\in N\cap M} |(d\alpha_i')_z|_{g'} \cdot \chi\left[\mathrm{dist}_{\,\hat{g}}\left(O, V_i^{(\nu)}\right)\right] \\
&\leq \sum_{\sigma\in S} \sum_{(i,\nu)\in N\cap M_\sigma} |(d\alpha_i')_z|_{g'} \cdot 4 \cdot \chi(r(x)) + \sum_{(i,\nu)\in N\cap M} |(d\alpha_i')_z|_{g'} \cdot 4 \cdot \chi(r(x)) \\
&\leq 4kA \cdot \chi(r(x)).
\end{aligned}$$

Similarly, for each point $z = \Psi(x)$ with $x \in W$ and each tangent vector $v \in T_z^{(1,0)}U'$, we have

$$|\mathcal{L}(\alpha')(v,v)| \leq 4kA \cdot |v|_{g'}^2 \cdot \chi(r(x)).$$

Setting $\beta' = \log \alpha'$, we get $\beta = \beta' \circ \Psi$ on W and, for each point $z = \Psi(x)$ with $x \in W$ and each tangent vector $v \in T_z^{(1,0)}U'$, we get

$$\begin{aligned}
\mathcal{L}(\beta')(v,v) &= \frac{1}{\alpha(x)} \cdot \mathcal{L}(\alpha')(v,v) - \frac{1}{(\alpha(x))^2} |\partial\alpha'(v)|^2 \\
&\geq -\frac{4kA \cdot \chi(r(x))}{\alpha(x)} \cdot |v|_{g'}^2 - \frac{16k^2A^2 \cdot (\chi(r(x)))^2}{(\alpha(x))^2} \cdot |v|_{g'}^2 \\
&\geq -\left(16kA + 256k^2A^2\right) \cdot |v|_{g'}^2.
\end{aligned}$$

Hence, whenever $U \cap D_\nu \neq \emptyset$, the function $\beta' + B\rho_\nu'$ will be strictly plurisubharmonic on a neighborhood of $\Psi(W)$ in U' and will satisfy $\beta + B\rho_\nu \circ \Upsilon = (\beta' + B\rho_\nu') \circ \Psi$ on W. Covering $\overline{\Omega}$ by finitely many such neighborhoods U, we see that we may choose a constant $R_0 > 0$ so that the property (6.1.3) holds for all covering spaces $\widehat{X}$ and all choices of Z.

We now modify β to get a function with values in $F \setminus N_{\mathbb{R}}(E;\eta)$ on Z. According to Lemma 6.2, for each $\sigma \in S$, we may choose a number $b_\sigma \in [0, 6\eta)$ such that

$$c_\sigma = \log\left(\chi\left[\mathrm{dist}_{\,\hat{g}}(O, Z_\sigma)\right]\right) + b_\sigma \in F \setminus N_{\mathbb{R}}(E;\eta).$$

We may then define a C^∞ function γ on $\widehat{X}$ by the locally finite sum

$$\gamma = \sum_{\sigma\in S} \sum_{(i,\nu)\in M_\sigma} \alpha_i^{(\nu)} \cdot b_\sigma.$$

Easier versions of the previous arguments show that $\gamma \equiv b_\sigma$ on $N_{\hat{g}}(Z;\delta) \subset \widehat{\Omega}$ for each $\sigma \in S$, and, for some constant $R_1 > 0$ independent of the choice of the covering $\widehat{X}$, the analytic set

Z, and the number η, the function $\gamma + R_1\eta \cdot \rho_\nu \circ \Upsilon$ will be C^∞ strictly plurisubharmonic on $\Upsilon^{-1}(D_\nu)$ for $\nu = 1, \dots, m$. Setting $R = \max(R_0, R_1)$, it now follows that any positive C^∞ exhaustion function τ on X which is equal to $\beta + \gamma$ on $\widehat{\Omega}$ has the required properties. $\square$

7. The main result

Theorem 0.1 is an immediate consequence of the following theorem which is proved in this section:

Theorem 7.1. *Let (X, g) be a connected reduced Hermitian complex space; let q be a positive integer; let Y be a compact analytic subset of dimension $\leq q$; let C and S be analytic subsets of Y such that C is a union of irreducible components of Y, $\dim S < q$, S contains Y_{sing} as well as every irreducible component of Y of dimension $< q$, and $Y \setminus (C \cup S)$ is Stein; let $\{D_\nu\}_{\nu=1}^m$ be relatively compact open subsets of X; let ρ_ν be a C^∞ strictly plurisubharmonic function on a neighborhood of $\overline{D}_\nu$ for each $\nu = 1, \dots, m$; and let $\epsilon > 0$. Then, for every choice of constants δ_1, δ_2, δ_3, and δ_4 with $\epsilon \gg \delta_4 \gg \delta_3 \gg \delta_2 \gg \delta_1 > 0$ (i.e. one must choose δ_4 sufficiently small relative to ϵ, δ_3 sufficiently small relative to δ_4, and so on), there exists a neighborhood Ω of Y in X such that, for every $\eta > 0$, for every set $E \subset \mathbb{R}$ with $|s - t| \geq 4\eta$ for all $s, t \in E$ with $s \neq t$, for every set $F \subset \mathbb{R}$ with $[0, \infty) \subset N_{\mathbb{R}}(F; \eta)$, for every connected covering space $\Upsilon \colon \widehat{X} \to X$, for every union Z of irreducible components of $\widehat{Y} = \Upsilon^{-1}(Y)$ which has only compact connected components and which contains $\widehat{C} = \Upsilon^{-1}(C)$ as well as every q-dimensional compact irreducible component of $\widehat{Y}$ (i.e. Z contains every compact irreducible component of $\widehat{Y}$ not contained in $\widehat{S} = \Upsilon^{-1}(S)$), and for every positive continuous function θ on $\widehat{X}$, there exists a positive C^∞ exhaustion function φ on $\widehat{X}$ with the following properties relative to the Hermitian metric $\hat{g} = \Upsilon^* g$, the sets $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$ and $\widehat{D}_\nu = \Upsilon^{-1}(D_\nu)$ for $\nu = 1, \dots, m$, and the functions $\hat{\rho}_\nu = \rho_\nu \circ \Upsilon \colon \widehat{D}_\nu \to \mathbb{R}$ for $\nu = 1, \dots, m$:*

(i) *On $N(Z; \delta_1) \cap \widehat{\Omega}$, φ is locally constant with*
$$G = \varphi\left(N(Z; \delta_1) \cap \widehat{\Omega}\right) = \varphi(Z) \subset F \setminus N_{\mathbb{R}}(E; \eta);$$
(ii) *On $N(\widehat{S}; \delta_3) \cap \widehat{\Omega}$, φ is of class $\mathcal{W}^\infty(\hat{g}, q)$;*
(iii) *On the set $\left\{ p \in N(\widehat{S}; \delta_3) \cap \widehat{\Omega} \mid \varphi(p) \notin G \right\}$, φ is of class $\mathcal{SP}^\infty(\hat{g}, q)$;*
(iv) *For each $\nu = 1, \dots, m$, the function $\varphi - \hat{\rho}_\nu$ is of class $\mathcal{SP}^\infty(\hat{g}, q)$ on a neighborhood of $N(\widehat{S}; \delta_3) \cap \widehat{D}_\nu \cap \widehat{\Omega} \setminus N(Z; \delta_2)$;*
(v) *For each $\nu = 1, \dots, m$, the function $\varphi - \hat{\rho}_\nu$ is C^∞ strongly q-convex on a neighborhood of $\widehat{D}_\nu \cap \widehat{\Omega} \setminus \left[N(\widehat{S}; \delta_2) \cup N(Z; \delta_1)\right]$;*

(vi) *On* $\widehat{\Omega} \setminus \left[N(\widehat{S};\delta_4) \cup N(Z;\delta_1)\right]$, $\varphi > \theta$;

(vii) *On* $\left\{ p \in \widehat{\Omega} \mid \varphi(p) \notin G \right\}$, φ *is* C^∞ *strongly* q*-convex; and*

(viii) *On* $\widehat{\Omega}$, φ *is of class* $\mathcal{W}^\infty(q)$.

Proof. We may assume without loss of generality that $N(Y;\epsilon) \Subset D_1 \cup \cdots \cup D_m$. Applying Proposition 6.1, we may also assume that, for some constant $R_0 > 1$, for every $\eta > 0$, for every set $E \subset \mathbb{R}$ with $|s-t| \geq 4\eta$ for all $s,t \in E$ with $s \neq t$, for every set $F \subset \mathbb{R}$ with $[0,\infty) \subset N_{\mathbb{R}}(F;\eta)$, for every connected covering space $\Upsilon\colon \widehat{X} \to X$, and for every analytic set Z which is equal to a union of irreducible components of $\widehat{Y} = \Upsilon^{-1}(Y)$ and which has only compact connected components, there exists a C^∞ positive exhaustion function τ on $\widehat{X}$ with the following properties:

(7.1.1) The function $\tau + R_0 \cdot (1+\eta) \cdot \rho_\nu \circ \Upsilon$ is C^∞ strictly plurisubharmonic on $\Upsilon^{-1}(D_\nu)$ for $\nu = 1, \dots, m$; and

(7.1.2) For $\hat{g} = \Upsilon^* g$, τ is locally constant on $N_{\hat{g}}(Z;\epsilon)$ and $\tau\left(N_{\hat{g}}(Z;\epsilon)\right) \subset F \setminus N_{\mathbb{R}}(E;\eta)$.

Note that, in the above, $\tau(N(Z;\epsilon)) = \tau(Z)$ since τ is constant on the connected neighborhood $N(Z';\epsilon)$ of each connected component Z' of Z and $N(Z;\epsilon)$ is equal to the union of all such neighborhoods.

According to Proposition 5.2, if we choose constants δ_1, δ_2, δ_3, and δ_4 with $\epsilon \gg \delta_4 \gg \delta_3 \gg \delta_2 \gg \delta_1 > 0$, then there exists a relatively compact neighborhood Ω of Y in $N(Y;\delta_1)$ such that, for every connected covering space $\Upsilon\colon \widehat{X} \to X$, for every analytic set Z which is equal to a union of irreducible components of $\widehat{Y} \equiv \Upsilon^{-1}(Y)$ and which contains $\widehat{C} = \Upsilon^{-1}(C)$ as well as every q-dimensional compact irreducible component of $\widehat{Y}$, and for every positive continuous function θ on $\widehat{X}$, there exists a nonnegative C^∞ function α on $\widehat{X}$ with the following properties relative to the Hermitian metric $\hat{g} = \Upsilon^* g$, the sets $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$, $\widehat{S} = \Upsilon^{-1}(S)$, and $\widehat{D}_\nu = \Upsilon(D_\nu)$ for $\nu = 1, \dots, m$, and the functions $\hat{\rho}_\nu = \rho_\nu \circ \Upsilon\colon \widehat{D}_\nu \to \mathbb{R}$ for $\nu = 1, \dots, m$:

(7.1.3) On $N(Z;\delta_1) \cap \widehat{\Omega}$, $\alpha \equiv 0$;

(7.1.4) On $\left(\widehat{\Omega} \setminus N(Z;\delta_2)\right) \cup \left(\widehat{\Omega} \setminus \left[N(\widehat{S};\delta_2) \cup N(Z;\delta_1)\right]\right)$, $\alpha > 0$;

(7.1.5) On $N(\widehat{S};\delta_3) \cap \widehat{\Omega}$, α is of class $\mathcal{W}^\infty(\hat{g}, q)$;

(7.1.6) On the set $\left\{ p \in N(\widehat{S};\delta_3) \cap \widehat{\Omega} \mid \alpha(p) > 0 \right\}$, α is of class $\mathcal{SP}^\infty(\hat{g}, q)$;

(7.1.7) For each $\nu = 1, \dots, m$, the function $\alpha - \hat{\rho}_\nu$ is of class $\mathcal{SP}^\infty(\hat{g}, q)$ on a neighborhood of $N(\widehat{S};\delta_3) \cap \widehat{D}_\nu \cap \widehat{\Omega} \setminus N(Z;\delta_2)$;

(7.1.8) For each $\nu = 1, \dots, m$, the function $\alpha - \hat{\rho}_\nu$ is C^∞ strongly q-convex on a neighborhood of $\widehat{D}_\nu \cap \widehat{\Omega} \setminus \left[N(\widehat{S}; \delta_2) \cup N(Z; \delta_1) \right]$;

(7.1.9) On $\widehat{\Omega} \setminus \left[N(\widehat{S}; \delta_4) \cup N(Z; \delta_1) \right]$, $\alpha > \theta$; and

(7.1.10) On $\left\{ p \in \widehat{\Omega} \mid \alpha(p) > 0 \right\}$, α is C^∞ strongly q-convex.

Suppose now that $\eta > 0$, $E \subset \mathbb{R}$ with $|s - t| \geq 4\eta$ for all $s, t \in E$ with $s \neq t$, $F \subset \mathbb{R}$ with $[0, \infty) \subset N_{\mathbb{R}}(F; \eta)$, $\Upsilon \colon \widehat{X} \to X$ is a connected covering space, Z is a union of irreducible components of $\widehat{Y} = \Upsilon^{-1}(Y)$ which has only compact connected components and which contains $\widehat{C} = \Upsilon^{-1}(C)$ as well as every q-dimensional compact irreducible component of $\widehat{Y}$, and θ is a positive continuous function on $\widehat{X}$. Setting $\hat{g} = \Upsilon^* g$, $\widehat{\Omega} = \Upsilon^{-1}(\Omega)$, $\widehat{S} = \Upsilon^{-1}(S)$, and $\widehat{D}_\nu = \Upsilon^{-1}(D_\nu)$ and $\hat{\rho}_\nu = \rho_\nu \circ \Upsilon \colon \widehat{D}_\nu \to \mathbb{R}$ for $\nu = 1, \dots, m$, we get on $\widehat{X}$ a positive C^∞ exhaustion function τ and a nonnegative C^∞ function α satisfying (7.1.1)–(7.1.10). It is then easy to check that the positive C^∞ exhaustion function

$$\varphi \equiv \tau + R_0 \cdot (2 + \eta) \cdot \alpha$$

has the required properties (i)–(viii). $\square$

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